NONSTANDARD QUANTUM GROUPS –
TWISTING CONSTRUCTIONS AND NONCOMMUTATIVE
DIFFERENTIAL GEOMETRY

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Nonstandard Quantum Groups—Twisting Constructions and Noncommutative Differential Geometry

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The date of my admission as a research student was October 1992.

To Ruth
and my Family

Contents

Abstract

The general subject of this thesis is quantum groups. The major original results are obtained in the particular areas of twisting constructions and noncommutative differential geometry.

Chapters 1 and 2 are intended to explain to the reader what are quantum groups. They are written in the form of a series of linked results and definitions. Chapter 1 reviews the theory of Lie algebras and Lie groups, focusing attention in particular on the classical Lie algebras and groups. Though none of the quoted results are due to the author, such a review, aimed specifically at setting up the paradigm which provides essential guidance in the theory of quantum groups, does not seem to have appeared already. In Chapter 2 the elements of the quantum group theory are recalled. Once again, almost none of the results are due to the author, though in Section 2.10, some results concerning the non-standard Jordanian group are presented, by way of a worked example, which have not been published.

Chapter 3 concerns twisting constructions. We introduce a new class of 2-cocycles defined explicitly on the generators of certain multiparameter standard quantum groups. These allow us, through the process of twisting the familiar standard quantum groups, to generate new as well as previously known examples of non-standard quantum groups. In particular we are able to construct generalisations of both the Cremmer-Gervais deformation of $SL(3)$ and the so called esoteric quantum groups of Fronsdal and Galindo in an explicit and straightforward manner.

In Chapter 4 we consider the differential calculus on Hopf algebras as introduced by Woronowicz. We classify all 4-dimensional first order bicovariant calculi on the Jordanian quantum group $GL_{h,g}(2)$ and all 3-dimensional first order bicovariant calculi on the Jordanian quantum group $SL_h(2)$. In both cases we assume that the bicovariant bimodules are generated as left modules by the differentials of the quantum group generators. It is found that there are 3 1-parameter families of 4-dimensional bicovariant first order calculi on $GL_{h,g}(2)$ and that there is a single, unique, 3-dimensional bicovariant calculus on $SL_h(2)$. This 3-dimensional calculus may be obtained through a classical-like reduction from any one of the three families of 4-dimensional calculi on $GL_{h,g}(2)$. Details of the higher order calculi and also the quantum Lie algebras are presented for all calculi. The quantum Lie algebra obtained from the bicovariant calculus on $SL_h(2)$ is shown to be isomorphic to the quantum Lie algebra we obtain as an ad-submodule within the Jordanian universal enveloping algebra $U_h(\mathfrak{sl}_2(\mathbb{C}))$ and also through a consideration of the decomposition of the tensor product of two copies of the deformed adjoint module. We also obtain the quantum Killing form for this quantum Lie algebra.

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Preface

The theory of quantum groups is a precise mathematical generalisation of many elements of classical Lie theory. However, approaching the subject for the first time one could be forgiven for being confused about precisely what classical object corresponds to a particular quantum object. This is probably most acute when it comes to the quantum versions of classical coordinate rings. The fundamental paper of Faddeev, Reshetikhin and Takhtajan[**84**] in which these objects were originally defined is, at least initially, the most accessible of the early papers on quantum groups. The paper is accessible because it describes an essentially linear algebraic construction for a pair of Hopf algebras starting from a matrix R with particular properties. In certain standard cases corresponding to the classical series of Lie groups $SL_n(\mathbb{C})$, $SO_n(\mathbb{C})$ and $Sp_{2n}(\mathbb{C})$ the construction is claimed to provide a deformation, or quantisation, of the Hopf algebra of 'functions on the classical Lie group' and dually, a quantisation of the universal enveloping algebra of the Lie algebra of the Lie group. In these cases the quantisation of the enveloping algebra is apparently the same as that obtained by Drinfeld and Jimbo. However, there are very many questions raised at this point.

1. What exactly is the classical Hopf algebra of 'functions on the classical Lie group'?
2. The FRT Hopf algebras corresponding to the Lie groups $SL_n(\mathbb{C})$, $SO_n(\mathbb{C})$ and $Sp_{2n}(\mathbb{C})$ are generated in each case by n^2 generators with relations which look very much like variations of the usual defining properties of matrices in these groups. Why should this be?
3. What is the classical relationship between the Hopf algebra of 'functions on the classical Lie group' and the universal enveloping algebra of the corresponding Lie algebra?
4. What is it reasonable to expect for the relationship between the FRT Hopf algebras and the Drinfeld-Jimbo quantisations of the corresponding enveloping algebras and what exactly has been proved about the actual relationship?

The answers to these questions are peculiarly difficult to extract from the literature and it is the purpose of Chapters 1 and 2 to collect and review the relevant results.

In Chapter 3 we focus our attention on the notion of 'twisting' already encountered in Chapter 2. It can be that a pair of quantum groups are both related to the same classical object and yet have radically different Hopf algebraic properties. Very often these quantum groups are not homomorphically related but rather are related through the process of twisting. Indeed associated with $SL_n(\mathbb{C})$ there are, aside from the standard quantum group of Drinfeld, quantum groups discovered by Fronsdal and Galindo [**41**], Cremmer and Gervais [**22**] and also some conjectured quantum groups of Gerstenhaber, Giaquinto and Schack [**44**]. Based on Drinfeld's fundamental Theorem 2.15.9 which establishes the existence of a kind of universal quantisation for a given Lie group it is one of the outstanding problems of quantum group theory to establish the twists which we *believe* relate these objects. We have obtained these twists for the Fronsdal-Galindo quantum groups, which includes as a special case the first non-trivial Cremmer-Gervais quantum group. Our construction yields other previously unknown quantum groups when used in conjunction with twists discovered by Engeldinger and Kempf [**37**]. The twisting construction for the other Cremmer-Gervais quantum groups is still an open problem as is the proof of the Gerstenhaber-Giaquinto-Schack conjecture.

Quantum groups are noncommutative generalisations of algebras of functions on Lie groups. Lie groups are of course differentiable manifolds and there is a well known 'equivalence' between the algebra of functions on a manifold and the manifold itself. Indeed a good deal of classical differential geometry can be written in terms of the commutative algebra of functions on a manifold. Noncommutative geometry as pioneered by Connes [**19**] is based on the idea that by studying the correct generalisations of these structures for noncommutative algebras we are doing geometry on some generalisation of the manifold itself. In classical differential geometry Lie groups occupy a position of essential importance. It is natural then to investigate the algebraic structures on quantum groups which provide the correct generalisation of differential geometry on Lie groups. Woronowicz [**104**] developed the relevant theory and it has been intensively studied ever since. A puzzling feature of these calculi soon emerged however. Except for the general linear groups, the calculi were not of the same dimension as their classical counterparts. This has led to some alternative constructions all of which have drawbacks. Very few studies of noncommutive geometry on non-standard quantum groups have been carried out. In Chapter 4 we provide a fairly complete treatment for the non-standard Jordanian quantum group. There we classify Woronowicz type calculi and find that in pleasant contrast to the standard case, there is a single, unique calculus and this has the classical dimension. Furthermore, we consider the analog of a Lie algebra which emerges from this calculus and show that it is isomorphic to that which emerges from a very different dual construction.

CHAPTER 1

The Classical Picture

1.1. Introduction

There is a classical paradigm which provides at the very least the best way of understanding the interrelations between the many different objects which are called quantum groups. More than this though, the classical theory provides the motivation for key definitions in the most unified and consistent approach to quantum groups, the elements of which are recalled in Chapter 2.

It is the purpose of the present chapter to recall and review the relevant results from the classical theory. We concentrate in particular on the Lie groups $SL_n(\mathbb{C})$, $SO_n(\mathbb{C})$ and $Sp_{2n}(\mathbb{C})$ and their Lie algebras. There is an issue regarding terminology which the reader should be aware of. Namely, that in Lie theory these Lie groups are called the classical Lie groups, this use of the word classical having nothing to do with the use of the word when distinguishing from *quantum* objects. From now on, in this chapter, the word classical has the original Lie theory meaning.

In relation to the questions suggested by the FRT paper, Sections 1.12, 1.16, 1.18 and 1.20 are particularly relevant. These sections answer Questions 1 and 3 of the Preface and also provide the relevant framework for the answers to Questions 2 and 4 which will be provided in Chapter 2.

None of the results quoted in this section are new. However we have not found any one source where these results are all presented together. References for this chapter are **[21]**, **[14]**, **[92]**, **[91]** **[16]**, **[101]**,**[43]** and **[82]**.

1.2. Lie algebras

Let us recall the basic definitions.

DEFINITION 1.2.1. A *Lie algebra*, $\mathfrak{g}$, over a field, k, is a k-module equipped with a bilinear operation, the *Lie bracket*, $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, satisfying

$$[x, y] = -[y, x], \tag{1.2.1}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \tag{1.2.2}$$

for all $x, y, z \in \mathfrak{g}$. The first of these, (1.2.1), expresses the antisymmetry of the Lie bracket, while (1.2.2) replaces the associativity condition of associative algebras and is called the *Jacobi identity.*

DEFINITION 1.2.2. A *Lie algebra map* ϕ between two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ is a vector space map such that $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{g}$.

We will only be concerned here with complex or real Lie algebras, in which cases we have $k = \mathbb{C}$ or $\mathbb{R}$ respectively.

Whenever we have an associative algebra A with a product $m : A \otimes A \to A$, we can define a Lie bracket on A according to $[a, b] = m(a \otimes b) - m(b \otimes a)$, for all $a, b \in A$. This bracket satisfies both (1.2.1) and (1.2.2) and therefore equips the vector space A with the structure of a Lie algebra which we shall denote $L(A)$.

A typical example of this construction is the following: For any finite-dimensional vector space V the linear transformations of V, $\mathfrak{gl}(V) = \{M : V \to V \mid M \text{ linear}\}$, form a Lie algebra. The bracket is defined by $[M, N] = M \circ N - N \circ M$ for all $M, N \in \mathfrak{gl}(V)$ where $\circ$ is the usual composition of maps. In particular, if we suppose V to be an n-dimensional complex vector space and choose some basis for it, then the linear transformations become complex valued $n \times n$ matrices, and the Lie algebra obtained by this construction is denoted $\mathfrak{gl}_n(\mathbb{C})$ (if we regard this Lie algebra as a real Lie algebra then we denote it $\mathfrak{gl}(n, \mathbb{C})$).

We can now define what we mean by a representation of a Lie algebra.

DEFINITION 1.2.3. A *representation* of a Lie algebra $\mathfrak{g}$ is a pair (ρ, V), where ρ is a Lie algebra map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. Equivalently, we can emphasise V and talk of a $\mathfrak{g}$-module V as a vector space upon which there is defined an action $\triangleright$ such that

$$[x, y] \triangleright v = x \triangleright (y \triangleright v) - y \triangleright (x \triangleright v), \tag{1.2.3}$$

for all $x, y \in \mathfrak{g}$ and any $v \in V$.

A *submodule* of a $\mathfrak{g}$-module V is a subspace $U \subseteq V$ such that $x \triangleright u \in U$ for all $x \in \mathfrak{g}$ and $u \in U$. We say that a $\mathfrak{g}$-module V is *simple* or *irreducible* if its only submodules are 0 and V. It is *semi-simple* or *completely reducible* if it is a direct sum of simple submodules.

A vector space map $\phi : V \to W$ between two $\mathfrak{g}$-modules V and W is a $\mathfrak{g}$-*module map* if $\phi(x \triangleright v) = x \triangleright \phi(v)$ for all $x \in \mathfrak{g}$ and $v \in V$. We say that V and W are *equivalent* $\mathfrak{g}$-modules if ϕ is also an isomorphism.

There is a useful result which is usually called Schur's lemma:

THEOREM 1.2.4. *A $\mathfrak{g}$-module map $\phi : V \to W$ between two irreducible $\mathfrak{g}$-modules is either 0 or an isomorphism. If $W = V$ then $\phi = \lambda\,\mathrm{id}$ for some $\lambda \in \mathbb{C}$.*

It follows from this that the irreducible representations of an Abelian Lie algebra are all 1-dimensional.

The structure of Lie algebras is worked out through a detailed analysis of the *adjoint representation*. This is the representation $(\mathrm{ad}, \mathfrak{g})$ where

$$\mathrm{ad}(x)(y) = [x, y] \tag{1.2.4}$$

for all $x, y \in \mathfrak{g}$. In this case the $\mathfrak{g}$-module is $\mathfrak{g}$ itself, the action is provided by the Lie bracket and condition (1.2.3) is just the Jacobi identity.

Of pivotal importance in the theory of Lie algebras is a certain bilinear form.

DEFINITION 1.2.5. The *Killing form*, $\mathfrak{B} : \mathfrak{g} \otimes \mathfrak{g} \to k$, is the symmetric k-bilinear form defined as

$$\mathfrak{B}(x, y) = \mathrm{Tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y)) \tag{1.2.5}$$

for all $x, y \in \mathfrak{g}$ where Tr denotes the trace.

The Killing form is ad-invariant, i.e. $\mathfrak{B}([x,y],z) + \mathfrak{B}(y,[x,z]) = 0$ for all $x, y \in \mathfrak{g}$.

Let us employ the notation

$$[\mathfrak{h}, \mathfrak{h}'] = \operatorname{span}_k\{[x,y] \mid x \in \mathfrak{h}, y \in \mathfrak{h}'\}, \tag{1.2.6}$$

where $\mathfrak{h}$ and $\mathfrak{h}'$ are subsets of a Lie algebra $\mathfrak{g}$ with the given bracket. Then we say that a Lie algebra $\mathfrak{g}$ is *Abelian* if $[\mathfrak{g}, \mathfrak{g}] = 0$. A subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a *Lie subalgebra* of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. If a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ also satisfies $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$ then it is called an *ideal* of $\mathfrak{g}$. However, only when $\mathfrak{h} \neq \mathfrak{g}$ and $\mathfrak{h} \neq 0$ do we regard $\mathfrak{h}$ as a *proper* ideal.

1.3. Complex simple Lie algebras

DEFINITION 1.3.1. A *complex simple Lie algebra* $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$ which is not Abelian and contains no proper ideals.

Such Lie algebras have the following important properties:

THEOREM 1.3.2. *If $\mathfrak{g}$ is a complex simple Lie algebra then the Killing form on $\mathfrak{g}$ is non-degenerate and every finite-dimensional $\mathfrak{g}$-module is completely reducible.*

A key ingredient in the structural analysis which leads to the classification of complex simple Lie algebras is the *Cartan subalgebra* — a maximal Abelian Lie subalgebra upon which the restriction of the adjoint representation is semi-simple.

THEOREM 1.3.3. *Every complex simple Lie algebra has at least one Cartan subalgebra. Given two Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$ of $\mathfrak{g}$ there is an automorphism σ of $\mathfrak{g}$ such that $\mathfrak{h}' = \sigma(\mathfrak{h})$.*

The dimensions of all Cartan subalgebras of a given Lie algebra are thus the same, l say, which we define to be the *rank* of $\mathfrak{g}$.

As the simple modules of Abelian Lie algebras are 1-dimensional it follows that having chosen a Cartan subalgebra $\mathfrak{h}$, the restriction of the adjoint representation to $\mathfrak{h}$ splits $\mathfrak{g}$ into a direct sum of 1-dimensional $\mathfrak{h}$-modules. It turns out that except for the l identical modules corresponding to the action of $\mathfrak{h}$ on $\mathfrak{h}$, these modules are mutually inequivalent. Therefore we can define, for some non-zero $\alpha \in \mathfrak{h}^*$ which will be called *roots*, the 1-dimensional *root subspaces* $\mathfrak{g}_\alpha$ of $\mathfrak{g}$ by

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \operatorname{ad}(h)(x) = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}. \tag{1.3.1}$$

Denote by Φ the set of all roots, then we have the *Cartan decomposition* of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha. \tag{1.3.2}$$

The roots actually span $\mathfrak{h}^*$ and to every $\alpha \in \Phi$ we also have $-\alpha \in \Phi$ but no other multiple of α. The restriction of the Killing form to $\mathfrak{h}$ is still non-degenerate, which allows us to introduce elements $h_\alpha \in \mathfrak{h}$ corresponding to each root according to $\mathfrak{B}(h_\alpha, h) = \alpha(h)$ for all $h \in \mathfrak{h}$. A non-degenerate symmetric bilinear form $\langle , \rangle : \mathfrak{h}^* \otimes \mathfrak{h}^* \to \mathbb{C}$ is then induced on $\mathfrak{h}^*$ by $\langle \alpha, \beta \rangle = \mathfrak{B}(h_\alpha, h_\beta)$ for all $\alpha, \beta \in \Phi$. In fact, this bilinear form is positive-definite on the real linear span of the roots $\mathfrak{h}^*_{\mathbb{R}}$ making $\mathfrak{h}^*_{\mathbb{R}}$ an l-dimensional Euclidian space. For each pair of roots $\{\alpha, -\alpha\}$, the single basis elements of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ may be chosen to be E_α and $E_{-\alpha}$ respectively such that $\mathfrak{B}(E_\alpha, E_{-\alpha}) = \frac{2}{\langle \alpha, \alpha \rangle}$. We may also define elements

$H_\alpha = \frac{2h_\alpha}{\langle\alpha,\alpha\rangle}$ for all $\alpha \in \Phi$ and note that they span $\mathfrak{h}$. The Lie brackets for the arbitrary complex simple Lie algebra may then be written in the following *Chevalley form*:

$$[H_\alpha, H_\beta] = 0, \tag{1.3.3}$$

$$[H_\alpha, E_\beta] = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}E_\beta, \tag{1.3.4}$$

$$[E_\alpha, E_{-\alpha}] = H_\alpha, \tag{1.3.5}$$

$$[E_\alpha, E_\beta] = \begin{cases} 0 & \text{if } \alpha+\beta \notin \Phi, \\ N_{\alpha,\beta}E_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi, \end{cases} \tag{1.3.6}$$

where $N_{\alpha,\beta} = \pm(p+1)$, p being the greatest integer such that $\beta - p\alpha \in \Phi$ (there is an algorithm for determining a consistent choice of signs of the $N_{\alpha,\beta}$ which is based only on knowledge of Φ).

Let us note that the numbers $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$ can be shown to be integers. Also, in the real Euclidian space spanned by the roots there are reflections s_α, for each $\alpha \in \Phi$, defined by

$$s_\alpha(\beta) = \beta - \frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha. \tag{1.3.7}$$

With multiplication provided by composition of maps, the set of these reflections generate a group, $\mathcal{W}$, called the *Weyl group*, which leaves Φ invariant.

The pair $(\Phi, \mathfrak{h}^*)$ is an example of a *reduced root system* in $\mathfrak{h}^*$. That is, Φ is a subset of $\mathfrak{h}^*$ spanning $\mathfrak{h}^*$ such that

1. Φ is finite but does not contain 0;
2. for any $\alpha \in \Phi$ there exists an element of $\mathfrak{h}$, namely H_α, such that $\alpha(H_\alpha) = 2$;
3. for any $\alpha \in \Phi$ the reflection s_α leaves Φ invariant;
4. for all $\alpha, \beta \in \Phi$, $\alpha(H_\beta) \in \mathbb{Z}$;
5. for each $\alpha \in \Phi$, α and $-\alpha$ are the only roots in Φ proportional to α.

Two root systems $(\Phi_1, \mathfrak{h}_1^*)$ and $(\Phi_2, \mathfrak{h}_2^*)$ are isomorphic if there is a vector space isomorphism $\phi : \mathfrak{h}_1^* \to \mathfrak{h}_2^*$ such that $\phi(\Phi_1) = \Phi_2$ and $\phi(\beta)(H_{\phi(\alpha)}) = \beta(H_\alpha)$ for all $\alpha, \beta \in \Phi_1$. It can be shown that the simplicity of $\mathfrak{g}$ means that the root system $(\Phi, \mathfrak{h}^*)$ is *irreducible* in the sense that we cannot write $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ and $\Phi = \Phi_1 \cup \Phi_2$ with $\Phi_1 \subset \mathfrak{h}_1^*$ and $\Phi_2 \subset \mathfrak{h}_2^*$ such that $(\Phi_1, \mathfrak{h}_1^*)$ and $(\Phi_2, \mathfrak{h}_2^*)$ are root systems.

A general property of root systems $(\Phi, \mathfrak{h}^*)$ is that there always exists a subset $\Pi = \{\alpha_i\}_{i=1}^l$ of Φ, called the set of *simple roots*, which form a basis of $\mathfrak{h}^*$ and in terms of which every other root can be written as a linear combination with either non-negative (in which case we say the root is *positive*) or non-positive (in which case we say the root is *negative*) integer coefficients. The set of all roots is therefore divided equally into two subsets, $\Phi = \Phi^+ \cup \Phi^-$, where Φ^+ and Φ^- denote the positive and negative roots respectively.

The Weyl group for a root system $(\Phi, \mathfrak{h}^*)$ is generated by the l reflections $\{s_{\alpha_i}\}_{i=1}^l$ and indeed *every* root in Φ may be obtained through a knowledge of the simple roots alone. Explicitly,

$$\Phi = \{w(\alpha_i) \mid w \in \mathcal{W} \text{ and } \alpha_i \in \Pi\}. \tag{1.3.8}$$

Thus, all the information needed to reconstruct the entire root system from the simple roots is encoded in the *Cartan matrix*, defined as the $l \times l$ matrix $\mathbf{A}$ with $A_{ij} = \frac{2\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}$.

The choice of Π does not effect the Cartan matrix up to similarity as it can be shown that any other basis of simple roots Π' for a root system $(\Phi, \mathfrak{h}^*)$ is such that $\Pi' = w(\Pi)$ where $w \in \mathcal{W}$ and it is a general property of the Weyl group that $\langle w(\alpha), w(\beta)\rangle = \langle \alpha, \beta\rangle$ for any $\alpha, \beta \in \Phi$ and $w \in \mathcal{W}$. It follows from this that the root systems reconstructed from Π and Π' are isomorphic.

In the particular case of the root system of a Lie algebra it can be shown that different choices of Cartan subalgebra lead to isomorphic root systems so we see that to any complex simple Lie algebra there corresponds an irreducible root system which is unique up to isomorphism and which is itself specified up to isomorphism by a Cartan matrix. Moreover it may be shown that two Lie algebras are isomorphic if and only if their corresponding root systems are isomorphic.

In the original Lie algebra we can define elements, $H_i = H_{\alpha_i}$, $X_i = E_{\alpha_i}$ and $Y_i = E_{-\alpha_i}$ for each $\alpha_i \in \Pi$. These elements actually already generate $\mathfrak{g}$ according to the following result of J. P. Serre.

THEOREM 1.3.4. *The complex simple Lie algebra $\mathfrak{g}$ with Cartan matrix $\mathbf{A}$ is isomorphic to the complex simple Lie algebra generated by the $3l$ generators $\{X_i, Y_i, H_i\}_{i=1}^{l}$ subject to the following relations:*

$$[H_i, H_j] = 0, \tag{1.3.9}$$

$$[X_i, Y_j] = \delta_{ij} H_i, \tag{1.3.10}$$

$$[H_i, X_j] = A_{ji} X_j, \quad [H_i, Y_j] = -A_{ji} Y_j, \tag{1.3.11}$$

$$\mathrm{ad}(X_i)^{-A_{ji}+1}(X_j) = 0, \quad \mathrm{ad}(Y_i)^{-A_{ji}+1}(Y_j) = 0, \tag{1.3.12}$$

where in the last line $i \neq j$ and $\mathrm{ad}(x)^n(y) = [x, [x \ldots [x, y] \ldots]]$ with n nested brackets.

In fact Serre's result is more general: *Given the Cartan matrix of any irreducible root system then Theorem 1.3.4 provides the construction of a complex simple Lie algebra whose root system is that specified by the Cartan matrix.*

The basis $\{H_i, E_\alpha, E_{-\alpha} \,|\, i = 1 \ldots l, \alpha \in \Phi\}$ of $\mathfrak{g}$ is called a *Cartan-Weyl basis* while the generators $\{H_i, X_i, Y_i \,|\, i = 1 \ldots l\}$ are called *Chevalley-Serre generators.*

We see that the classification of the complex simple Lie algebras is equivalent to the classification of irreducible reduced root systems and that this classification may be carried out through a determination of the possible Cartan matrices.

THEOREM 1.3.5. *There are four infinite series of finite complex simple Lie algebras, denoted by A_l, B_{l+1}, C_{l+2} and D_{l+3} respectively, with $l \geq 1$, together with five exceptional cases which are denoted by E_6, E_7, E_8, F_4 and G_2. The Cartan matrices are presented, for example, in* [**21**].

The four infinite series of Lie algebras are the *classical complex Lie algebras*, so called because it can be shown that $A_l \cong \mathfrak{sl}_{l+1}(\mathbb{C})$, $B_l \cong \mathfrak{so}_{2l+1}(\mathbb{C})$, $C_l \cong \mathfrak{sp}_{2l}(\mathbb{C})$ and $D_l \cong \mathfrak{so}_{2l}(\mathbb{C})$ where $\mathfrak{sl}_{l+1}(\mathbb{C})$, $\mathfrak{so}_{2l+1}(\mathbb{C})$, $\mathfrak{sp}_{2l}(\mathbb{C})$, and $\mathfrak{so}_{2l}(\mathbb{C})$ are the matrix Lie algebras which arise as the Lie algebras of the classical complex Lie groups (see Section 1.14). They are Lie subalgebras of $\mathfrak{gl}_n(\mathbb{C})$ for particular n. Indeed $\mathfrak{sl}_{l+1}(\mathbb{C})$ is the subspace of $(l+1)\times(l+1)$ traceless matrices. The orthogonal Lie algebras $\mathfrak{so}_{2l+1}(\mathbb{C})$ and $\mathfrak{so}_{2l}(\mathbb{C})$ may be regarded as consisting of skew-symmetric matrices. Equivalently, we can regard $\mathfrak{so}_{2l+1}(\mathbb{C})$ as consisting of matrices $\mathbf{M}$ satisfying $\mathbf{M}^t\mathbf{L} + \mathbf{L}\mathbf{M} = 0$ where

$$\mathbf{L} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_l & \mathbf{0} \\ \mathbf{I}_l & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix}, \tag{1.3.13}$$

$\mathbf{I}_l$ indicates the $l \times l$ identity matrix and the superscript t indicates the transpose. Similarly, we may regard $\mathfrak{so}_{2l}(\mathbb{C})$ as the Lie algebra of matrices $\mathbf{M}$ satisfying $\mathbf{M}^t\mathbf{S} + \mathbf{S}\mathbf{M} = 0$ where

$$\mathbf{S} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_l \\ \mathbf{I}_l & \mathbf{0} \end{pmatrix}. \tag{1.3.14}$$

The symplectic Lie algebra $\mathfrak{sp}_{2l}(\mathbb{C})$ consists of matrices $\mathbf{M}$ satisfying $\mathbf{M}^t\mathbf{J} + \mathbf{J}\mathbf{M} = 0$ where

$$\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_l \\ -\mathbf{I}_l & \mathbf{0} \end{pmatrix}. \tag{1.3.15}$$

EXAMPLE 1.3.6. A_1 is the complex simple Lie algebra with basis elements denoted $\{X, Y, H\}$ say, upon which the Lie bracket is defined by

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H. \tag{1.3.16}$$

However, we may just as well think of A_1 as the matrix Lie algebra with basis $\{\mathbf{H}, \mathbf{X}, \mathbf{Y}\}$, say, where

$$\mathbf{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1.3.17}$$

These matrices can also be regarded as providing a faithful (injective) representation (ρ, V) of the abstract Lie algebra A_1 on a 2-dimensional complex vector space V with $\rho(H) = \mathbf{H}$, $\rho(X) = \mathbf{X}$ and $\rho(Y) = \mathbf{Y}$.

For a given complex simple Lie algebra $\mathfrak{g}$ in the Cartan-Weyl basis we can identify many Lie subalgebras. The Cartan subalgebra $\mathfrak{h}$, with basis $\{H_i\}_{i=1}^n$, has already been introduced. We also have two 'nilpotent' Lie subalgebras $\mathfrak{n}^+$ and $\mathfrak{n}^-$, with bases $\{E_\alpha \mid \alpha \in \Phi^+\}$ and $\{E_{-\alpha} \mid \alpha \in \Phi^+\}$ respectively. Thus, as a vector space the Lie algebra has the *triangular decomposition* $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. The *Borel subalgebras* are defined as $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ and $\mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-$. Also, for every positive root α there is an $\mathfrak{sl}_2(\mathbb{C})$*-subalgebra* $\mathfrak{sl}_2(\mathbb{C})_\alpha$ with basis $\{H_\alpha, E_\alpha, E_{-\alpha}\}$.

1.4. The universal enveloping algebra

Starting with an abstract Lie algebra $\mathfrak{g}$ we may associate with it a certain unital associative algebra.

DEFINITION 1.4.1. The *universal enveloping algebra* $U(\mathfrak{g})$ is defined to be the quotient of $T(\mathfrak{g})$ (the tensor algebra built from $\mathfrak{g}$) by the two-sided ideal in $T(\mathfrak{g})$ generated by all elements of the form $[W, W'] - W \otimes W' + W' \otimes W$ with $W, W' \in \mathfrak{g}$. That is,

$$U(\mathfrak{g}) = T(\mathfrak{g}) / \langle [W, W'] - W \otimes W' + W' \otimes W \rangle, \tag{1.4.1}$$

for all $W, W' \in \mathfrak{g}$.

Denoting by π the canonical projection from $T(\mathfrak{g})$ to $U(\mathfrak{g})$ and by ι the canonical embedding of $\mathfrak{g}$ in $T(\mathfrak{g})$, the composition $\pi \circ \iota$ is a Lie algebra map from $\mathfrak{g}$ to $U(\mathfrak{g})$. The adjective 'universal' refers to the following property of $U(\mathfrak{g})$: Given any associative algebra, A, and a Lie algebra map $f : \mathfrak{g} \to L(A)$ there exists a unique algebra map, $\phi : U(\mathfrak{g}) \to A$ such that $\phi \circ (\pi \circ \iota) = f$. This universal property means that the representation theories of $\mathfrak{g}$ and $U(\mathfrak{g})$ are equivalent.

EXAMPLE 1.4.2. $U(\mathfrak{sl}_2(\mathbb{C}))$ is the algebra over $\mathbb{C}$ generated by the elements $\{X, Y, H\}$ subject to the commutator relations

$$HX - XH = 2X, \quad HY - YH = -2Y, \quad XY - YX = H. \tag{1.4.2}$$

(We have suppressed the tensor product here.)

In this example the monomials $\{Y^\alpha H^\beta X^\gamma : \alpha, \beta, \gamma \in \mathbb{N}\}$ form a basis for $U(\mathfrak{sl}_2(\mathbb{C}))$. This basis is a particular example of the following general result for universal enveloping algebras of Lie algebras called the Poincaré-Birkhoff-Witt (PBW) theorem.

THEOREM 1.4.3. *If a finite dimensional Lie algebra, $\mathfrak{g}$, has a basis, $\{x_i\}_{i=1}^n$ say, then the set of elements $\{x_1^{i_1} x_2^{i_2} \dots x_n^{i_n} \mid i_j \in \mathbb{N}\}$ form a basis of $U(\mathfrak{g})$.*

Corresponding to the triangular decomposition of $\mathfrak{g}$, we have the vector space isomorphism, $U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+)$.

1.5. Representation theory of complex simple Lie algebras

Roots are a key concept in the classification of complex simple Lie algebras. They are particular examples of a more general concept which is essential for the classification of the irreducible representations of those Lie algebras. We assume our representations to be finite-dimensional and take as understood the equivalence of representations of $\mathfrak{g}$ and $U(\mathfrak{g})$.

DEFINITION 1.5.1. Suppose V is a $\mathfrak{g}$-module and $\lambda \in \mathfrak{h}^*$, then define V^λ by,

$$V^\lambda = \{v \in V \mid h \triangleright v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}. \tag{1.5.1}$$

When $V^\lambda \neq 0$ we call it the *weight space* consisting of *weight vectors* corresponding to the *weight*, λ, of V. We say that λ has *multiplicity* equal to the dimension of V^λ.

It turns out that $\mathfrak{h}$ can be taken to act diagonally on any finite dimensional $\mathfrak{g}$-module so that $V = \oplus_\lambda V^\lambda$. Also, given $v \in V^\lambda$, then $(E_\alpha \triangleright v) \in V^{\lambda+\alpha}$ for all $\alpha \in \Phi$. In an obvious notation we could also write this as $\mathfrak{g}_\alpha \triangleright V^\lambda \subset V^{\lambda+\alpha}$.

It can be shown that every finite-dimensional $\mathfrak{g}$-module contains a certain distinguished kind of weight vector v of weight λ say, called a *singular vector*, such that $\mathfrak{n}^+ \triangleright v = 0$. Then v generates a submodule $W = U(\mathfrak{g}) \triangleright v$ of V spanned by vectors of the form $E_{-\beta_1}^{m_1} E_{-\beta_2}^{m_2} \dots E_{-\beta_k}^{m_k} \triangleright v$, where v is being acted upon by elements of the PBW-basis of $U(\mathfrak{n}^-)$ (i.e. $\beta_i \in \Phi^+$ and $m_i \in \mathbb{N}$). This means that the weights of W are of the form $\lambda - \sum_{i=1}^l n_i \alpha_i$ where $n_i \in \mathbb{N}$ and the α_i are the simple roots. Furthermore we see that λ has multiplicity 1 as a weight of W from which it follows that W is indecomposable and therefore irreducible.

DEFINITION 1.5.2. A singular vector v of weight Λ of a $\mathfrak{g}$-module V which is such that $U(\mathfrak{g}) \triangleright v = V$ is called a *highest weight vector* and V is then called a *highest weight module* with highest weight Λ and written $V(\Lambda)$.

DEFINITION 1.5.3. The l fundamental weights of a Lie algebra $\mathfrak{g}$ are defined as the elements $\Lambda_{(1)}, \Lambda_{(2)}, \ldots, \Lambda_{(l)} \in \mathfrak{h}^*$ such that $\Lambda_{(i)}(H_j) = \delta_{ij}$ for $1 \leq i, j \leq l$.

The fundamental weights (also called the coroots) provide another basis of $\mathfrak{h}^*$, related to the basis of simple roots by $\Lambda_{(i)} = \sum_{j=1}^{l} (\mathbf{A}^{-1})_{ji} \alpha_j$. The subset of $\mathfrak{h}^*$, $P^+ = \sum_{i=1}^{l} \mathbb{N}\Lambda_{(i)}$, is then called the set of *dominant integral weights,* while $P = \sum_{i=1}^{l} \mathbb{Z}\Lambda_{(i)}$ is the *weight lattice.* The root lattice is defined as $Q = \sum_{i=1}^{l} \mathbb{Z}\alpha_i$, and it is clear that $Q \subset P \subset \mathfrak{h}^*$.

The basic theorem classifying finite-dimensional, irreducible $\mathfrak{g}$-modules may now be stated.

THEOREM 1.5.4. *If V is an irreducible $\mathfrak{g}$-module then it contains a highest weight vector, v_Λ, say, of weight Λ, which is unique up to scalar multiplication and such that $\Lambda \in P^+$. Conversely, to every $\Lambda \in P^+$ there is an irreducible finite dimensional $\mathfrak{g}$-module $V(\Lambda)$ which is unique up to isomorphism and has highest weight Λ.*

EXAMPLE 1.5.5. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, with basis $\{H, X, Y\}$, has rank one so its irreducible representations may be denoted by $V(n) = V(n\Lambda_{(1)})$, with $n \in \mathbb{N}$. Each $V(n)$ has dimension $n+1$ and a basis may be chosen for it, $\{v_i^n\}_{i=0}^{n}$, such that the actions of the basis elements of $\mathfrak{sl}_2(\mathbb{C})$, are

$$H \triangleright v_i^n = (n - 2i) v_i^n, \tag{1.5.2}$$

$$X \triangleright v_i^n = (n - i + 1) v_{i-1}^n, \tag{1.5.3}$$

$$Y \triangleright v_i^n = (i+1) v_{i+1}^n. \tag{1.5.4}$$

In the physics literature, a different basis, e_m^j, where $m = -j, -j+1, \ldots, j-1, j$ is used, with the representations being labelled now by the numbers j such that $2j = n$

$$X \triangleright e_m^j = \sqrt{(j-m)(j+m+1)}\, e_{m+1}^j, \tag{1.5.5}$$

$$H \triangleright e_m^j = 2m e_m^j, \tag{1.5.6}$$

$$Y \triangleright e_m^j = \sqrt{(j+m)(j-m+1)}\, e_{m-1}^j. \tag{1.5.7}$$

Two irreducible modules, $V(1)$ and $V(2)$, are particularly important. The defining module $V(1)$ is the faithful representation which gives the realisation, which we saw in Example 1.3.6, of $\mathfrak{sl}_2(\mathbb{C})$ as an algebra of 2×2 complex traceless matrices. $V(2)$ is the adjoint module.

Also for the other classical complex simple Lie algebras, the first fundamental representations, $V(\Lambda_{(1)})$, are faithful representations. They are equivalent to the realisations of the abstract Lie algebras in terms of matrices $\mathbf{M}$ which satisfy $\mathrm{Tr}(\mathbf{M}) = 0$ in the case of $\mathfrak{sl}_{l+1}(\mathbb{C})$ and $\mathbf{M}^t\mathbf{X} + \mathbf{X}\mathbf{M} = 0$ with $\mathbf{X} = \mathbf{L}, \mathbf{S}, \mathbf{J}$ in the cases $\mathfrak{so}_{2l+1}(\mathbb{C})$, $\mathfrak{so}_{2l}(\mathbb{C})$ and $\mathfrak{sp}_{2l}(\mathbb{C})$ respectively.

1.6. Hopf algebras

We recall the definition of a Hopf algebra.

DEFINITION 1.6.1. A Hopf algebra is a sextuple $(A, m, \eta, \Delta, \epsilon, S)$ such that A is a $\mathbb{C}$-module (a vector space over $\mathbb{C}$) with $\mathbb{C}$-linear maps $m : A \otimes A \to A$ — the multiplication, $\eta : \mathbb{C} \to A$ — the unit, $\Delta : A \to A \otimes A$ — the coproduct, $\epsilon : A \to \mathbb{C}$ — the counit and $S : A \to A$ — the antipode, satisfying the following axioms:

$$m \circ (\mathrm{id} \otimes m) = m \circ (m \otimes \mathrm{id}), \tag{1.6.1}$$

$$m \circ (\mathrm{id} \otimes \eta) = \mathrm{id} = m \circ (\eta \otimes \mathrm{id}), \tag{1.6.2}$$

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta, \tag{1.6.3}$$

$$(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta, \tag{1.6.4}$$

$$m \circ (\mathrm{id} \otimes S) \circ \Delta = \eta \circ \epsilon = m \circ (S \otimes \mathrm{id}) \circ \Delta. \tag{1.6.5}$$

It is further required that Δ and ϵ be algebra maps, which immediately implies that m and η are coalgebra maps. The definition of a *bialgebra* is just that of a Hopf algebra without an antipode.

It follows from the definition that S is both an antialgebra map and an anti-coalgebra map. That is, $S(ab) = S(b)S(a)$ and $S(a_{(1)}) \otimes S(a_{(2)}) = S(a)_{(2)} \otimes S(a)_{(1)}$, where in the second equation we have introduced a version of the *Sweedler notation for coproducts* **[98]** in which the summation is suppressed. This notation is extremely useful and will be used from now on. It amounts to writing $\Delta(a) = a_{(1)} \otimes a_{(2)}$ with the understanding that the right hand side consists of a summation of the form $\sum_i a_{i(1)} \otimes a_{i(2)}$.

A Hopf algebra, A, is said to be *commutative* if it is commutative as an algebra. It is said to be *cocommutative* if $a_{(1)} \otimes a_{(2)} = a_{(2)} \otimes a_{(1)}$ for all $a \in A$.

DEFINITION 1.6.2. Two Hopf algebras, U and A, are said to be *Hopf algebras in non-degenerate duality*, if there exists a bilinear map, $\langle\, , \rangle : U \otimes A \to \mathbb{C}$, called a pairing, such that

$$\langle xy, a\rangle = \langle x, a_{(1)}\rangle\langle y, a_{(2)}\rangle, \quad \langle x, ab\rangle = \langle x_{(1)}, a\rangle\langle x_{(2)}, b\rangle, \tag{1.6.6}$$

$$\langle 1, a\rangle = \epsilon(a), \quad \langle x, 1\rangle = \epsilon(x), \quad \langle S(x), a\rangle = \langle x, S(a)\rangle, \tag{1.6.7}$$

which is non-degenerate in the sense that the algebra maps $\iota_U : U \to A^*$ and $\iota_A : A \to U^*$, given by $\iota_U(x)(a) = \langle x, a\rangle$ and $\iota_A(a)(x) = \langle x, a\rangle$ respectively, are injective. In this case the maps ι_U and ι_A constitute embeddings of U and A in A^* and U^* respectively. A possibly-degenerate pairing between Hopf algebras is called simply a *Hopf pairing*.

1.7. Representations of Hopf algebras

When we talk about a representation of a Hopf algebra, we mean a representation of its algebra sector. We consider only finite-dimensional representations.

DEFINITION 1.7.1. If A is a Hopf algebra, then a *left A-module* is a pair, $(\triangleright, V)$, where V is a vector space upon which there is an action, $\triangleright$, of A such that

$$(ab) \triangleright v = a \triangleright (b \triangleright v), \quad 1_A \triangleright v = v, \tag{1.7.1}$$

for all $a, b \in A$ and $v \in V$. Equivalently, a *representation* is a pair (ρ, V) where $\rho : A \to \mathfrak{gl}(V)$ is such that

$$\rho(ab)v = \rho(a)\rho(b)v, \tag{1.7.2}$$

for all $a, b \in A$ and $v \in V$.

EXAMPLE 1.7.2. An important example of a left A-module is provided by the *left adjoint action* of A on A defined as

$$a \triangleright b = a_{(1)} b S(a_{(2)}) \tag{1.7.3}$$

for all $a, b \in A$.

A vector space $U \subseteq V$ is called an A-submodule if $a \triangleright U \subseteq U$ for all $a \in A$. Reducibility, complete reducibility and irreducibility are then all defined in the obvious way. A map $\phi : V \to W$ between two A-modules is called an A-module map if $\phi(a \triangleright v) = a \triangleright \phi(v)$ for all $a \in A$. We have the following 'Schur's lemma' result:

THEOREM 1.7.3. *An A-module map $\phi : V \to W$ between two irreducible A-modules is either 0 or an isomorphism. If $W = V$ then $\phi = \lambda\,\mathrm{id}$ for some $\lambda \in \mathbb{C}$.*

A consequence of this result is that the irreducible representations of any commutative Hopf algebra are 1-dimensional.

If we choose a basis, $\{v_i\}_{i=1}^n$ say, for the vector space, V, of a representation, (ρ, V), then we obtain a matrix representation, $\rho(a)$, for all $a \in A$ according to, $a \triangleright v_i = \sum_{j=1}^n \rho_{ji}(a) v_j$, where $\rho_{ij}(a) = (\rho(a))_{ij}$.

DEFINITION 1.7.4. The *matrix coefficients* of a left A-module, V, are the linear forms $\rho^V_{\alpha,v} : A \to \mathbb{C}$ defined by $\rho^V_{\alpha,v}(a) = \alpha(a \triangleright v)$ for all $\alpha \in V^*$ and $v \in V$. The space spanned by the matrix coefficients of *all* finite-dimensional representations of a Hopf algebra, A, is denoted A° and called the Hopf dual of A.

In all the examples in which we are interested, the Hopf algebra, A say, is infinite dimensional. In this case the dual maps $m^*, \eta^*, \Delta^*, \epsilon^*$ and S^* do *not* provide a Hopf structure on A^*. The problem with A^* is that the dual of the multiplication map, $m^* : A^* \to (A \otimes A)^*$ is in general taking us out of $A^* \otimes A^*$, as $(A \otimes A)^*$ is only isomorphic to $A^* \otimes A^*$ when A is finite and in general is 'bigger' than $A^* \otimes A^*$. The importance of the Hopf dual stems from the fact that the restrictions of the dual maps *do* define a Hopf algebra structure on A°.

The restriction of Δ^* to $A^* \otimes A^*$ together with ϵ^* certainly provides an algebra structure on A^*. That this algebra structure closes on A° is guaranteed by certain natural A-module constructions. Indeed, if V and W are A-modules, then so is their direct sum, $V \oplus W$, with the action of A given by $a \triangleright (v \oplus w) = (a \triangleright v) \oplus (a \triangleright w)$. This provides an addition for A° — for all $\alpha \in V^*$, $\beta \in W^*$, $v \in V$ and $w \in W$ we have $\rho^V_{\alpha,v} + \rho^W_{\beta,w} = \rho^{V \oplus W}_{\alpha \oplus \beta, v \oplus w}$. Scalar multiplication in A° follows from the fact that any constant multiple, cV, of an A-module V is naturally an A-module, so $c\rho^V_{\alpha,v} = \rho^V_{c\alpha,v} = \rho^V_{\alpha,cv}$. The Hopf structure of A allows us to construct the tensor product A-module, $V \otimes W$, given two A-modules, V and W, with the action provided by the coproduct as $a \triangleright (v \otimes w) = (a_{(1)} \triangleright v) \otimes (a_{(2)} \triangleright w)$ for any $a \in A$ and all $v \in V$ and $w \in W$. This provides a multiplication for matrix coefficients, $\rho^V_{\alpha,v} \rho^W_{\beta,w} = \rho^{V \otimes W}_{\alpha \otimes \beta, v \otimes w}$. The counit of A provides the trivial 1-dimensional representation of A, (ϵ, V_ϵ), with $a \triangleright v_\epsilon = \epsilon(a) v_\epsilon$ for any $a \in A$ and v_ϵ the single basis element of V_ϵ. Denoting by α_ϵ the single basis element of V^*_ϵ such that $\alpha_\epsilon(v_\epsilon) = 1$, then a unit for A° is just $\rho^{V_\epsilon}_{\alpha_\epsilon, v_\epsilon}$. It is straightforward to check that the multiplication and unit we have defined for A° are simply the restrictions of the dual maps $\Delta^* : (A \otimes A)^* \to A^*$ and $\epsilon^* : \mathbb{C} \to A^*$ (to be more precise, $\epsilon^*(1)(a) = \epsilon(a) = \rho_{\alpha_\epsilon, v_\epsilon}$).

The details of the full Hopf structure on A° are presented in the following theorem.

THEOREM 1.7.5. *A° is a Hopf algebra. Indeed, for all $v \in V$, $w \in W$, $\alpha \in V^*$ and $\beta \in W^*$, where V and W are any pair of left A-modules, we have*

$$\rho^V_{\alpha,v} + \rho^W_{\beta,w} = \rho^{V\oplus W}_{\alpha\oplus\beta,v\oplus w}, \tag{1.7.4}$$

$$c\rho^V_{\alpha,v} = \rho^V_{c\alpha,v} = \rho^V_{\alpha,cv}, \tag{1.7.5}$$

where $c \in \mathbb{C}$, and

$$\begin{aligned} m(\rho^V_{\alpha,v} \otimes \rho^W_{\beta,w}) &= \rho^{V\otimes W}_{\alpha\otimes\beta,v\otimes w}, \\ \eta(1) &= \epsilon_A, \\ \Delta(\rho^V_{\alpha,v}) &= \sum_i \rho^V_{\alpha,v_i} \otimes \rho^V_{\alpha_i,v}, \\ \epsilon(\rho^V_{\alpha,v}) &= \alpha(v), \\ S(\rho^V_{\alpha,v}) &= \rho^{V^*}_{v,\alpha}, \end{aligned} \tag{1.7.6}$$

where $\{v_i\}$ and $\{\alpha_i\}$ are dual bases for V and V^* respectively and ϵ_A is the counit in A.

REMARK 1.7.6. To understand the expression for the antipode of A° we need to recall that if V is an A-module then so is V^* with the action defined as $(a \triangleright \alpha)(v) = \alpha(S(a) \triangleright v)$.

REMARK 1.7.7. Let us also note here that if $U \subset V$ is a submodule of V then the matrix coefficients of U are obtained as the subset of matrix coefficients $\rho^V_{\alpha,v}$ of V such that $v \in U$ and $\alpha \in U^*$. Also, we can form the quotient space, V/U, which is again an A-module, now with action $a \triangleright (v + U) = a \triangleright v + U$. The matrix coefficients of V/U are the subset of matrix coefficients, $\rho_{\alpha,v}$, of V for which $v \notin U$ and $\alpha \in U^\perp$ where $U^\perp$ is the annihilator of U.

For any given n-dimensional representation, (ρ, V), choosing a basis for V provides us with a matrix representation and particular examples of matrix coefficients, the *matrix elements*, $\{\rho^V_{ij}\}^n_{i,j=1}$, of the representation. The space spanned by the matrix elements of (ρ, V) is independent of the particular choice of basis and they span the space of matrix coefficients of V.

1.8. Corepresentations of Hopf algebras

We may also consider 'representations' of the coalgebra sector.

DEFINITION 1.8.1. A *co-representation* is a pair, (Δ_V, V), where $\Delta_V : V \to V \otimes A$ is a linear map such that

$$(\Delta_V \otimes \mathrm{id}) \circ \Delta_V = (\mathrm{id} \otimes \Delta) \circ \Delta_V, \quad (\mathrm{id} \otimes \epsilon) \circ \Delta_V = \mathrm{id}\,. \tag{1.8.1}$$

We say that V is a *right A-comodule*, with Δ_V a *right coaction*.

EXAMPLE 1.8.2. An important example of a corepresentation of A is provided by the the *right adjoint coaction*, Ad^*_R, which is defined on any $a \in A$ as

$$Ad^*_R(a) = a_{(2)} \otimes S(a_{(1)})a_{(3)} \tag{1.8.2}$$

A vector space $U \subseteq V$ such that $\Delta_V(U) \subseteq U \otimes A$ is called an A-subcomodule. With this notion in place, the definitions of reducibility, complete reducibility and irreducibility are all obvious. A map $\phi : V \to W$ between two A-comodules is called an A-comodule map if $(\phi \otimes \mathrm{id}) \circ \Delta_V = \Delta_W \circ \phi$. Once again we have a 'Schur's lemma' result.

THEOREM 1.8.3. *An A-comodule map $\phi : V \to W$ between two irreducible A-comodules is either 0 or an isomorphism. If $W = V$ then $\phi = \lambda\,\mathrm{id}$ for some $\lambda \in \mathbb{C}$.*

It follows from this result is that the irreducible corepresentations of any cocommutative Hopf algebra are 1-dimensional.

DEFINITION 1.8.4. The *matrix coefficients* of a co-representation, (Δ_V, V), of a Hopf algebra A are the elements $\pi^V_{\alpha,v} \in A$ for any $v \in V$ and $\alpha \in V^*$ such that

$$\pi^V_{\alpha,v} = (\alpha \otimes \mathrm{id}) \circ \Delta_V. \tag{1.8.3}$$

There is a Sweedleresque notation for coactions, namely $\Delta_V(v) = v_{(V)} \otimes v_{(A)}$ for all $v \in V$.

We have constructions of comodules analogous to those for modules. Given two A-comodules, V and W, with respective coactions, Δ_V and Δ_W, their direct sum is also an A-comodule with coaction $\Delta_{V\oplus W}(v \oplus w) = \Delta_V(v) + \Delta_W(w)$ and so is their tensor product, with coaction $\Delta_{V\otimes W}(v \otimes w) = v_{(V)} \otimes w_{(W)} \otimes v_{(A)}w_{(A)}$ for all $v \in V$ and $w \in W$. Once again any constant multiple, cV, of an A-comodule is again an A-comodule, and so is the dual vector space, V^*, with coaction, $\Delta_{V^*}(\alpha) = \alpha_{(V^*)} \otimes \alpha_{(A)}$, such that for any $\alpha \in V^*$

$$\alpha_{(V^*)}(v)\alpha_{(A)} = \alpha(v_{(V)})S(v_{(A)}) \tag{1.8.4}$$

for all $v \in V$. The trivial 1-dimensional representation this time originates from the unit of A; we denote it by $(\Delta_{V_\eta}, V_\eta)$. Taking its single basis element to be v_η with $\alpha_\eta \in V^*_\eta$ such that $\alpha_\eta(v_\eta) = 1$ we have $\Delta_{V_\eta}(v_\eta) = v_\eta \otimes \eta(1)$. Thus the matrix coefficient is just $\pi_{\alpha_\eta, v_\eta} = 1_A$ where 1_A is the unit in A.

Once again there is a Hopf algebra structure on the space of matrix coefficients of all finite dimensional co-representations. This Hopf structure is formally precisely that in Theorem 1.7.5, but with ρ replaced by π and $\eta(1) = 1_A$.

Fixing a particular basis for V, $\{v_i\}_{i=1}^n$ say, there is a uniquely determined family of elements of A, $\{\pi^V_{ij}\}_{i,j=1}^n$ such that $\Delta_V(v_i) = \sum_{j=1}^n v_j \otimes \pi^V_{ji}$. The π^V_{ij} are called the matrix elements of the corepresentation (Δ_V, V) and span its space of matrix coefficients.

REMARK 1.8.5. In fact any Hopf algebra is spanned by the matrix coefficients of its finite dimensional corepresentations.

1.9. The correspondence between representations and corepresentations for non-degenerately paired Hopf algebras

If U and A are Hopf algebras in non-degenerate duality, suppose (Δ_V, V) is a corepresentation of A, then choosing a basis for V, $\{v_i\}$, we can write $\Delta_V(v_i) = \sum_j v_j \otimes \pi^V_{ji}$. The π^V_{ij} may be identified through the non-degenerate duality with elements of U^* and since $\Delta(\pi^V_{ik}) = \sum_j \pi_{ij} \otimes \pi_{jk}$ we can define a representation of U, (ρ, V), according to $\rho(X)v_i = \sum_j \pi_{ji}(x)v_j$ for all $X \in U$. Moreover it is clear that (Δ_V, V) is uniquely specified by this representation. If $U(V)$ denotes the subspace of U^* spanned by the matrix coefficients of the U-module V and $A(V)$ denotes the subspace of A spanned by the matrix coefficients of the A-comodule V, then $U(V) = A(V) \subset A$. In the other direction, starting with a representation (ρ, V) of U, $\rho(X)v_i = \sum_j \rho^V_{ji}(x)v_j$, we can not in general associate a corepresentation of A as there is no guaranteed inclusion of U^* in A. Thus we have a one-to-one correspondence between representations (ρ, V) of U and comodules (Δ_V, V) of A *for all U-modules such that $U(V) \subseteq A$.* This correspondence preserves all the usual representation attributes such as complete reducibility and irreducibility.

1.10. Enveloping algebras are Hopf algebras

For any complex simple Lie algebra, $\mathfrak{g}$, the enveloping algebra, $U(\mathfrak{g})$, is a Hopf algebra. Here is the precise result.

THEOREM 1.10.1. *$U(\mathfrak{g})$ is a Hopf algebra with the Hopf maps defined on the generators as*

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \epsilon(x) = 0, \quad S(x) = -x, \tag{1.10.1}$$

for $x \in \mathfrak{g}$, and extended linearly to the whole of $U(\mathfrak{g})$ as algebra, algebra and anti-algebra maps respectively. Moreover, $U(\mathfrak{g})$ is cocommutative, *that is, $x_{(1)} \otimes x_{(2)} = x_{(2)} \otimes x_{(1)}$ for all $x \in \mathfrak{g}$.*

We may regard $\mathfrak{g}$ as the subspace of *primitive* elements of $U(\mathfrak{g})$, i.e. those elements upon which the coproduct has the form in (1.10.1).

EXAMPLE 1.10.2. The adjoint representation of a Lie algebra, $\mathfrak{g}$, is the restriction to $\mathfrak{g}$ of the left adjoint action of $U(\mathfrak{g})$ on $U(\mathfrak{g})$

$$x \triangleright y = x_{(1)} y S(x_{(2)}) = xy - yx. \tag{1.10.2}$$

As $U(\mathfrak{g})$ is cocommutative, all its irreducible corepresentations are 1-dimensional. The irreducible representations of $U(\mathfrak{g})$ were classified in Section 1.5.

The constructions of the previous two sections can now be applied to $U(\mathfrak{g})$-modules. Thus if (ρ^V, V) and (ρ^W, W) are representations of $\mathfrak{g}$ then so is $(\rho^{V\otimes W}, V \otimes W)$ with

$$\rho^{V\otimes W}(x)(v \otimes w) = \rho^V(x)v \otimes w + v \otimes \rho^W(x)w, \tag{1.10.3}$$

for all $v \in V$, $w \in W$ and $x \in U(\mathfrak{g})$. The trivial representation of $\mathfrak{g}$ is just the 1-dimensional 'highest-weight' module, $V(0)$, such that $x \triangleright v = 0$ for all $x \in U(\mathfrak{g})$ and $v \in V$. If V is a representation of $\mathfrak{g}$ then so is V^* with $(x \triangleright \alpha)(v) = \alpha(-x \triangleright v)$.

We will be particularly interested in the Hopf dual of the enveloping algebra, $U(\mathfrak{g})^\circ$. There is a natural Hopf algebra pairing between $U(\mathfrak{g})$ and $U(\mathfrak{g})^\circ$. In fact this pairing renders $U(\mathfrak{g})$ and $U(\mathfrak{g})^\circ$ Hopf algebras in *non-degenerate duality.* On the one hand we have the natural embedding of $U(\mathfrak{g})^\circ$ in $U(\mathfrak{g})^*$. Dually we therefore have a linear map $\iota : U(\mathfrak{g}) \to (U(\mathfrak{g})^\circ)^*$ and this should be injective. If it were not then for all finite $U(\mathfrak{g})$-modules V we would have some non-zero $x \in U(\mathfrak{g})$ such that $\rho_{\alpha,v}(x) = 0$ for any $v \in V$, $\alpha \in V^*$. On the contrary, we have the following result due to Harish-Chandra:

THEOREM 1.10.3. *For any non-zero element, x, of the enveloping algebra, $U(\mathfrak{g})$, of a complex simple Lie algebra, $\mathfrak{g}$, there exists a finite dimensional representation of $\mathfrak{g}$, (ρ, V), such that $\rho(x) \neq 0$. Furthermore, in the cases of $\mathfrak{g}$ a classical complex simple Lie algebra, this representation is constructed as a tensorial power of the defining representation.*

We can now deduce immediately that the representation theory of $U(\mathfrak{g})$ is equivalent to the corepresentation theory of $U(\mathfrak{g})^\circ$.

1.11. (Co)Module morphisms and matrix coefficient relations

Suppose A is a Hopf algebra and V and W are two finite-dimensional A-modules with respective bases $\{v_i\}_{i=1}^n$ and $\{w_i\}_{i=1}^m$ such that their matrix elements are ρ_{ij}^V and ρ_{ij}^W respectively. If $\phi : V \to W$ is an A-module map such that $\phi(v_i) = \sum_{j=1}^m \phi_{ji} v_j$ then the matrix elements satisfy

$$\sum_{j=1}^{n} \phi_{ij}\rho_{jk}^V = \sum_{j=1}^{m} \rho_{ij}^W \phi_{jk}. \tag{1.11.1}$$

Similarly, if V and W are A-comodules and $\phi : V \to W$ is an A-comodule map, then the matrix elements of the respective corepresentations, π_{ij}^V and π_{ij}^W, satisfy (1.11.1) but with ρ replaced by π.

If V and W are any two $U(\mathfrak{g})$-modules, then $V\otimes W$ and $W\otimes V$ are isomorphic as $U(\mathfrak{g})$-modules. The isomorphism holds because they are isomorphic as vector spaces and there is a natural $U(\mathfrak{g})$-module map, $P_{V,W} : V \otimes W \to W \otimes V$, such that $P_{V,W}(v \otimes w) = w \otimes v$ for all $v \in V$ and $w \in W$. That $P_{V,W}$ is a Hopf algebra map follows immediately from the cocommutativity of $U(\mathfrak{g})$. It is natural to investigate the situation for a general Hopf algebra, A. Thus, we are interested in invertible A-module maps, $\Phi^{V,W} : V\otimes W \to W\otimes V$ for all finite-dimensional A-modules V, W. Notice that if $\Phi^{V,W}$ is such a map, and with bases chosen as before

$$\Phi^{V,W}(v_i \otimes w_j) = \sum_{k=1}^{m}\sum_{l=1}^{n} \Phi_{kl,ij}^{V,W} w_k \otimes v_l, \tag{1.11.2}$$

then the matrix elements of the respective representations, ρ_{ij}^V and ρ_{ij}^W, satisfy the following relations

$$\sum_{k=1}^{n}\sum_{l=1}^{m} \Phi_{ij,kl}^{V,W} \rho_{ks}^V \rho_{lt}^W = \sum_{k=1}^{m}\sum_{l=1}^{n} \rho_{ik}^W \rho_{jl}^V \Phi_{kl,st}^{V,W}. \tag{1.11.3}$$

Similarly, if V and W are A-comodules and $\Phi^{V,W} : V\otimes W \to W\otimes V$ is an A-comodule map, then the matrix elements of the respective corepresentations, π_{ij}^V and π_{ij}^W satisfy (1.11.3) but with ρ replaced by π.

1.12. Tensor product decompositions of the defining representations of the classical complex simple Lie algebras

In this section we restrict our discussion to the classical complex simple Lie algebras, A_l, B_l, C_l and D_l.

The $\mathfrak{g}$-modules $V(\Lambda_{(i)})$ for $i = 1 \ldots l$ are called the fundamental modules. The first fundamental representation of a classical complex simple Lie algebra, $\mathfrak{g}$, has highest weight $\Lambda_{(1)}$ but we will denote the corresponding module simply by V. We will also call this the defining representation because viewed as matrix representations these representations provide realisations of the Lie algebras A_l, B_l, C_l and D_l as the matrix Lie algebras $\mathfrak{sl}_{l+1}(\mathbb{C})$, $\mathfrak{so}_{2l+1}(\mathbb{C})$, $\mathfrak{sp}_{2l}(\mathbb{C})$ and $\mathfrak{so}_{2l}(\mathbb{C})$ respectively (they are faithful representations).

Given the fundamental modules all other irreducible $\mathfrak{g}$-modules can be obtained as submodules of their tensor products. Indeed, if $\Lambda = n_1\Lambda_{(1)} + n_2\Lambda_{(2)} + \ldots + n_l\Lambda_{(l)}$ then

$V(\Lambda)$ is a submodule of

$$\underbrace{V(\Lambda_{(1)}) \otimes \ldots V(\Lambda_{(1)})}_{n_1} \otimes \ldots \otimes \underbrace{V(\Lambda_{(l)}) \otimes \ldots V(\Lambda_{(l)})}_{n_l} \tag{1.12.1}$$

It follows that the Hopf dual $U(\mathfrak{g})^{\circ}$ of an enveloping algebra is generated by the matrix coefficients of the fundamental modules.

Let us denote by $\wedge^i(V)$ the image of $T^i(V)$ in the exterior algebra $T(V)/I$ where I is the two sided ideal in $T(V)$ generated by all elements of the form $v \otimes v$.

EXAMPLE 1.12.1. In the case of $\mathfrak{sl}_2(\mathbb{C})$-modules there is a particularly neat result. In this case the defining module, V, is the only fundamental module, and we have $V(n) := V(n\Lambda_{(1)}) = \mathrm{Sym}^n(V)$ where $\mathrm{Sym}^n(V)$ is the image of $T^n(V)$ in the symmetric algebra $T(V)/I$ where I is the two sided ideal in $T(V)$ generated by all elements of the form $v \otimes v' - v' \otimes v$. Moreover there is a simple expression, called the *Clebsch-Gordan series,* for the decomposition of the tensor product of two irreducible $\mathfrak{g}$-modules, $V(n) \otimes V(m)$, into a direct sum of irreducibles. In terms of the j-labels of physics, we have

$$V(2j_1) \otimes V(2j_2) \cong \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} V(2j). \tag{1.12.2}$$

For the basis, e^j_m, where $m = -j, -j+1, \ldots, j-1, j$, this isomorphism corresponds to a change of basis from the unreduced basis $\{e^{j_1}_{m_1} \otimes e^{j_2}_{m_2}\}$, to a reduced basis, $e^{(j_1 j_2)j}_m$, according to

$$e^{(j_1 j_2)j}_m = \sum_{m_1,m_2} C^{j_1,j_2,j}_{m_1,m_2,m} e^{j_1}_{m_1} \otimes e^{j_2}_{m_2}, \tag{1.12.3}$$

where $C^{j_1,j_2,j}_{m_1,m_2,m}$ are *Clebsch-Gordan coefficients.*

In fact for the classical Lie algebras most of the fundamental modules can be obtained as submodules of tensor products of just the defining modules. To be precise, this important result is as follows.

THEOREM 1.12.2. *For the Lie algebras A_l with $l \geq 1$ all irreducible representations of A_l can be obtained as submodules of tensor products of the defining representation, and in fact*

$$\wedge^i(V) = V(\Lambda_{(i)}) \tag{1.12.4}$$

for each $i = 1 \ldots l$.

For the Lie algebras B_l with $l \geq 2$ we cannot obtain the module $V(n\Lambda_{(l)})$, where n is odd, as a submodule in any iterated tensor products of V, only irreducible representations whose highest weight contains even multiples of $\Lambda_{(l)}$. In this case

$$\wedge^i(V) = V(\Lambda_{(i)}) \tag{1.12.5}$$

for each $i = 1 \ldots l-1$ but

$$\wedge^l(V) = V(2\Lambda_{(l)}). \tag{1.12.6}$$

For the Lie algebras C_l with $l \geq 3$ each fundamental representation can be obtained in iterated tensor products of the defining representation. In this case the modules $\wedge^i(V)$ are *not irreducible for $i \geq 2$. In fact we have*

$$\wedge^i(V) = \bigoplus_{j\geq 0} V(\Lambda_{(i-2j)}), \tag{1.12.7}$$

for each $i = 1 \dots l$.

For the Lie algebras D_l with $l \geq 4$ we cannot obtain $V(\Lambda_{(l-1)})$ or $V(\Lambda_{(l)})$ as submodules in iterated tensor products of the defining representation. Rather, we are only able to recover irreducible representations whose whose highest weight contains $m\Lambda_{(l-1)} + n\Lambda_{(l)}$ where $m + n$ is even. We have

$$\wedge^i(V) = V(\Lambda_{(i)}), \tag{1.12.8}$$

for each $i = 1 \dots l-2$ and

$$\wedge^{l-1}(V) = V(\Lambda_{(l-1)} + \Lambda_{(l)}), \quad \wedge^l(V) = V(2\Lambda_{(l-1)}) \bigoplus V(2\Lambda_{(l)}). \tag{1.12.9}$$

We will also need the following result concerning dual representations:

THEOREM 1.12.3. *For the complex simple Lie algebras B_l ($l \geq 2$), C_l ($l \geq 3$), D_l ($l \geq 4$ and* even*), the representation $V(\Lambda)^*$ is equivalent to $V(\Lambda)$. In the case of A_l ($l \geq 1$), $V(n_1\Lambda_{(1)} + n_2\Lambda_{(2)} + \cdots + n_l\Lambda_{(l)})^*$ is equivalent to $V(n_l\Lambda_{(1)} + n_{l-1}\Lambda_{(2)} + \cdots + n_1\Lambda_{(l)})$ while for D_l ($l \geq 5$ and* odd*), $V(n_1\Lambda_{(1)} + n_2\Lambda_{(2)} + \cdots + n_l\Lambda_{(l)})^*$ is equivalent to $V(n_1\Lambda_{(1)} + n_2\Lambda_{(2)} + \cdots + n_{l-2}\Lambda_{(l-2)} + n_l\Lambda_{(l-1)} + n_{l-1}\Lambda_{(l)})^*$.*

The next result can be seen as a corollary of the last two results and our previous discussion of matrix coefficients and the Hopf dual.

THEOREM 1.12.4. *For each of the classical complex simple Lie algebras, $\mathfrak{sl}_{l+1}(\mathbb{C})$, $\mathfrak{so}_{2l+1}(\mathbb{C})$, $\mathfrak{sp}_{2l}(\mathbb{C})$ and $\mathfrak{so}_{2l}(\mathbb{C})$ the matrix elements of the defining representation generate a sub-Hopf algebra, $U(\mathfrak{g})^{(\circ)}$, of the Hopf dual, $U(\mathfrak{g})^\circ$, of the respective enveloping algebra. In fact $U(\mathfrak{sl}_{l+1}(\mathbb{C}))^{(\circ)} \cong U(\mathfrak{sl}_{l+1}(\mathbb{C}))^\circ$ and $U(\mathfrak{sp}_{2l}(\mathbb{C}))^{(\circ)} \cong U(\mathfrak{sp}_{2l}(\mathbb{C}))^\circ$ while $U(\mathfrak{so}_{2l+1}(\mathbb{C}))^{(\circ)}$ and $U(\mathfrak{so}_{2l}(\mathbb{C}))^{(\circ)}$ are proper sub-Hopf algebras of $U(\mathfrak{so}_{2l+1}(\mathbb{C}))^\circ$ and $U(\mathfrak{so}_{2l}(\mathbb{C}))^\circ$ respectively.*

REMARK 1.12.5. The point is that in any of the cases, $\mathfrak{sl}_{l+1}(\mathbb{C})$, $\mathfrak{so}_{2l+1}(\mathbb{C})$, $\mathfrak{sp}_{2l}(\mathbb{C})$ and $\mathfrak{so}_{2l}(\mathbb{C})$, the algebra generated by the matrix elements of the defining representation contains the matrix coefficients of all tensor powers of V. In the case of $\mathfrak{sl}_{l+1}(\mathbb{C})$ and $\mathfrak{sp}_{2l}(\mathbb{C})$ each simple module is a direct summand of such a tensor power so the matrix coefficients of *all* simple modules are in this algebra and since we dont get any new matrix coefficients by taking dual representations or direct sum representations $U(\mathfrak{g})^{(\circ)} = U(\mathfrak{g})^\circ$. In the case of $\mathfrak{so}_{2l+1}(\mathbb{C})$ and $\mathfrak{so}_{2l}(\mathbb{C})$ it is no longer the case that every simple module appears as a direct summand of tensor powers of the defining representation. However the simple modules which *do* appear (Theorem 1.12.2) *are* closed under the formation of dual representations (as follows from Theorem 1.12.3). Thus in these cases we have genuine Hopf subalgebras $U(\mathfrak{so}_{2l+1}(\mathbb{C}))^{(\circ)}$ and $U(\mathfrak{so}_{2l}(\mathbb{C}))^{(\circ)}$ of the respective Hopf duals.

Reasoning as for $U(\mathfrak{g})^\circ$, we conclude that $U(\mathfrak{g})$ and $U(\mathfrak{g})^{(\circ)}$ are in non-degenerate duality. In the cases of $\mathfrak{g} = \mathfrak{sl}_{l+1}(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{sp}_{2l}(\mathbb{C})$ *every* finite-dimensional representation of $U(\mathfrak{g})$ corresponds to a unique corepresentation of $U(\mathfrak{g})^{(\circ)}$ and vice-versa. However for the orthogonal Lie algebras, $\mathfrak{g} = \mathfrak{so}_{2l+1}(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{so}_{2l}(\mathbb{C})$ only those representations of $U(\mathfrak{g})$

which appear in tensor products of the defining representation have corresponding corepresentations, while it is still true that every corepresentation has a unique representation corresponding to it.

EXAMPLE 1.12.6. Let us consider the particular example of $U(\mathfrak{sl}_2(\mathbb{C}))^{(\circ)}$, denoting the matrix elements of the defining representation, V, with respect to the basis v_0^1, v_1^1 presented in Example 1.5.5, by $\rho_{00}, \rho_{01}, \rho_{10}, \rho_{11}$. Then the Hopf algebra $U(\mathfrak{sl}_2(\mathbb{C}))^{(\circ)}$ is generated by these matrix elements and the coalgebra structure is given by $\Delta(\rho_{ik}) = \sum_{j=0}^{1} \rho_{ij} \otimes \rho_{jk}$ and $\epsilon(\rho_{ij}) = \delta_{ij}$. Since $U(\mathfrak{sl}_2(\mathbb{C}))$ is cocommutative, $U(\mathfrak{sl}_2(\mathbb{C}))^{(\circ)}$ is certainly commutative and its unit is provided by the counit of $U(\mathfrak{sl}_2(\mathbb{C}))$. In the context of $U(\mathfrak{sl}_2(\mathbb{C}))^{(\circ)}$ we'll just denote it by $\mathbf{1}$. The second exterior power of V, $\wedge^2(V)$, is of course a 1-dimensional $U(\mathfrak{sl}_2(\mathbb{C}))$-module (the action provided by the usual action on $V \otimes V$) and as such we know that it is equivalent to the trivial $U(\mathfrak{sl}_2(\mathbb{C}))$-module provided the counit of $U(\mathfrak{sl}_2(\mathbb{C}))$. Thus, we have a $U(\mathfrak{sl}_2(\mathbb{C}))$-module map $\Phi : V \wedge V \to V(0)$ such that $\Phi(v_0 \wedge v_1) = \kappa v_0^0$ where v_0^0 is the single basis element of $V(0)$ and κ is a constant. But

$$\begin{aligned}
\Phi(x \triangleright v_0 \wedge v_1) &= \Phi(x \triangleright (v_0 \otimes v_1 - v_1 \otimes v_0)) && (1.12.10)\\
&= \Phi((\rho_{00}\rho_{11} - \rho_{01}\rho_{10})(x)v_0 \otimes v_1 - (\rho_{00}\rho_{11} - \rho_{01}\rho_{10})(x)v_1 \otimes v_0) && (1.12.11)\\
&= (\rho_{00}\rho_{11} - \rho_{01}\rho_{10})(x)\Phi(v_0 \wedge v_1) && (1.12.12)\\
&= \kappa(\rho_{00}\rho_{11} - \rho_{01}\rho_{10})(x)v_0^0, && (1.12.13)
\end{aligned}$$

and

$$x \triangleright \Phi(v_0 \wedge v_1) = \kappa \mathbf{1}(x) v_0^0, \tag{1.12.14}$$

for all $x \in U(\mathfrak{sl}_2(\mathbb{C}))$ from which we deduce the 'determinant' relation

$$\rho_{00}\rho_{11} - \rho_{01}\rho_{10} = \mathbf{1}. \tag{1.12.15}$$

For the antipode, observe from Theorem 1.12.3 that $V \cong V^*$ as $U(\mathfrak{sl}_2(\mathbb{C}))$-modules so that there must exist an invertible $U(\mathfrak{sl}_2(\mathbb{C}))$-module map $\phi : V \to V^*$. It is straightforward to establish that with respect to the given basis and its dual, we can take ϕ to be the matrix

$$\phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.12.16}$$

But we know from (1.11.1) that the matrix elements must then satisfy

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \begin{pmatrix} S(\rho_{00}) & S(\rho_{10}) \\ S(\rho_{01}) & S(\rho_{11}) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.12.17}$$

so that

$$S \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \begin{pmatrix} \rho_{11} & -\rho_{01} \\ -\rho_{10} & \rho_{00} \end{pmatrix}. \tag{1.12.18}$$

The techniques described in this example generalise for $U(\mathfrak{g})^{(\circ)}$. In particular, we obtain a 'determinant' relation in each case by considering the natural action on the single basis element of the top exterior power of the defining representation. The antipode for $U(\mathfrak{sl}_{l+1}(\mathbb{C}))^{(\circ)}$ is obtained from the relations resulting from the $U(\mathfrak{g})$-module map corresponding to the isomorphism, $V(\Lambda_{(1)})^* \cong \wedge^l V(\Lambda_{(1)})$. In the other cases we use the maps corresponding to $V(\Lambda_{(1)})^* \cong V(\Lambda_{(1)})$. In these cases we actually obtain further relations, namely

$$\rho^t \mathbf{X} \rho \mathbf{X}^{-1} = \mathbf{I}, \tag{1.12.19}$$

where we are using an obvious matrix notation, and $\mathbf{X} = \mathbf{L}, \mathbf{S}, \mathbf{J}$ in the cases $\mathfrak{so}_{2l+1}(\mathbb{C})$, $\mathfrak{so}_{2l}(\mathbb{C})$ and $\mathfrak{sp}_{2l}(\mathbb{C})$ respectively. Note that with a different choice of basis for $V(\Lambda_{(1)})$ the matrix elements are changed to some linear combination of the ρ_{ij} and the matrices $\mathbf{X}$ will be different also. Indeed, if in the cases of $\mathfrak{so}_{2l+1}(\mathbb{C})$ and $\mathfrak{so}_{2l}(\mathbb{C})$ we took, as we certainly could, a basis of $V(\Lambda_{(1)})$ in terms of which the Lie algebras were realised as skew-symmetric matrices, then the matrix $\mathbf{X}$ in these cases would have been simply the identity and the extra relations would have looked like

$$\varrho \varrho^t = \mathbf{I}, \tag{1.12.20}$$

where ϱ_{ij} are the matrix elements corresponding to the alternative choice of basis.

1.13. Lie groups

We begin with the basic definitions.

DEFINITION 1.13.1. A group G which is also a differentiable (infinitely differentiable) manifold is a *Lie group* if the operations of multiplication and taking inverses are differentiable as maps from $G \times G$ to G and G to G respectively.

In particular, the left and right translation maps, $L_g : G \to G$ and $R_g : G \to G$ given respectively by $L_g(h) = gh$ and $R_g(h) = hg$ are differentiable diffeomorphisms.

DEFINITION 1.13.2. A differentiable map $\phi : G \to G'$ between two Lie groups G and G' which is also a group homomorphism is called a *Lie group homomorphism.*

DEFINITION 1.13.3. The *tangent space* at any point, g, of G is denoted $T_g(G)$. It is the real vector space of point derivations of germs of real valued functions at g.

Recall that the dimension of a differentiable manifold is the dimension of the Euclidian space which locally approximates the manifold. This is also the dimension of the tangent space $T_g(G)$ at any point $g \in G$.

If v is an element of $T_g(G)$ and f is a real-valued differentiable function on G then on some chart, (U_g, ψ_g), about g, whose coordinate functions are $\{x_i\}_{i=1}^d$ where d is the dimension of G, we may write

$$v(f) = \sum_{i=1}^{d} v_i \left. \frac{\partial \overline{f}}{\partial x_i} \right|_{\psi_g(g)}, \tag{1.13.1}$$

where $\overline{f} = f \circ \psi_g$. Abbreviating, an arbitrary tangent vector at g may be written uniquely as $v = \sum_{i=1}^d \partial / \partial x_i|_g$.

DEFINITION 1.13.4. A *differentiable vector field* on G is a differentiable assignment. $X : g \to X|_g$, of a tangent vector, $X|_g$, to any point $g \in G$.

On a chart (U_g, ψ_g), we may express an arbitrary vector field, X, uniquely as

$$X|_U = \sum_{i=1}^{d} \zeta_i \frac{\partial}{\partial x_i}\bigg|_{U_g}, \tag{1.13.2}$$

where ζ_i are differentiable functions on U_g. If X and Y are two differentiable vector fields then their *Lie bracket*, $[X, Y]$, is also a differentiable vector field.

DEFINITION 1.13.5. If $\phi : G \to G'$ is a differentiable map then there is an induced map $\mathrm{d}\phi$, the *differential* of ϕ, such that for all $g \in G$, $\mathrm{d}\phi : T_g(G) \to T_{\phi(g)}(G')$ according to

$$(\mathrm{d}\phi(v))(f) = v(f \circ \phi), \tag{1.13.3}$$

for any $v \in T_g(G)$.

DEFINITION 1.13.6. A vector field, X, on a Lie group G is called *left-invariant* if and only if for all $g \in G$ and any $h \in G$

$$X|_{gh} = \mathrm{d}L_g(X|_h). \tag{1.13.4}$$

The real vector space of all left-invariant vector fields is denoted $\Xi_L(G)$.

It may readily be confirmed that the Lie bracket of any two left-invariant vector fields is again left-invariant. The following important theorem tells us that the vector space of all left-invariant vector fields is d-dimensional.

THEOREM 1.13.7. *To any tangent vector $v \in T_e(G)$, where $e \in G$ is the identity of the group, may be associated a left-invariant vector field, X^v, such that $X^v|_g = \mathrm{d}L_g(v)$. The map $v \to X^v$ is a vector space isomorphism between $T_e(G)$ and $\Xi_L(G)$.*

DEFINITION 1.13.8. The Lie algebra of a Lie group G is the vector space $T_e(G)$ equipped with the Lie bracket, $[v, w] = [X^v, X^w]|_I$ for all $v, w \in T_e(G)$. It is denoted by $\mathfrak{g}$.

Any vector field, X, may be expressed as a sum of the form $X = \sum_{i=1}^d \zeta_i X^{v_i}$ where the ζ_i are differentiable functions on G and X^{v_i} is the basis of left invariant vector fields corresponding to a basis $\{v_i\}_{i=1}^d$ of $T_e(G)$.

DEFINITION 1.13.9. A differentiable curve $\sigma : (a, b) \subset \mathbb{R} \to G$ is an *integral curve of a vector field X centred at the identity* if $\sigma(0) = I$ $(0 \in (a, b))$ and $\mathrm{d}\sigma(\mathrm{d}/\mathrm{d}t)|_{t=s\in(a,b)} = X|_{\sigma(s)}$.

For a general differentiable manifold, $\mathcal{M}$, it is true that to any differentiable vector field X and for any point P in $\mathcal{M}$, there exists a unique integral curve of X, $\sigma_P^X(t)$, centred at the point P and defined on some interval (a, b) containing 0 such that if s, t and $s + t$ are all in (a, b) then $\sigma^X_{\sigma_P^X(s)}(t) = \sigma_P^X(s + t)$. In the particular case of Lie groups we have the following key result:

THEOREM 1.13.10. *To every left-invariant vector field X of G there is a unique integral curve centred at the identity with the whole of $\mathbb{R}$ as its domain.*

If such an integral curve is denoted $\sigma^v(t)$ where $v \in T_e(G)$ is such that $X|_I = v$ then $\sigma^v : \mathbb{R} \to G$ is a Lie group homomorphism called a *one-parameter subgroup*. There is a 1-1 correspondence between $T_e(G)$ and one-parameter subgroups.

DEFINITION 1.13.11. The exponential map $\exp : \mathfrak{g} \to G$ is defined as

$$\exp(tv) = \sigma^v(t), \tag{1.13.5}$$

for any $v \in \mathfrak{g}$ and $t \in \mathbb{R}$.

THEOREM 1.13.12. *The exponential map* $\exp : \mathfrak{g} \to G$ *is differentiable and its differential,* $\mathrm{d}\exp : T_0(\mathfrak{g}) \to T_e(G)$*, is the identity map. There is an open neighbourhood,* U_0*, of* $\mathfrak{g}$ *about* 0*, and an open neighbourhood,* U_e*, of* G *about* e *such that* $\exp : U_0 \to U_e$ *is a diffeomorphism.*

EXAMPLE 1.13.13. $GL(n,\mathbb{R})$ is a subset of $M(n,\mathbb{R}) \cong \mathbb{R}^{n^2}$, the set of all real matrices, and $GL(n,\mathbb{C})$ is a subset of $M(n,\mathbb{C}) \cong \mathbb{R}^{2n^2}$, the set of all complex matrices. Let us just discuss $GL(n,\mathbb{R})$ and note that an analogous discussion holds for $GL(n,\mathbb{C})$. $M(n,\mathbb{R})$ is a differentiable n^2-dimensional manifold with a global coordinate system in terms of the single chart, $(M(n,\mathbb{R}), \mathrm{Id})$ where $\mathrm{Id}(\mathrm{M}) = (M_{11}, M_{12}, \ldots, M_{1n}, \ldots, M_{n1}, \ldots, M_{nn})$ for any matrix $\mathrm{M} \in M(n,\mathbb{R})$. The coordinate functions of this chart are the maps $x_{ij} : M(n,\mathbb{R}) \to \mathbb{R}$ where $x_{ij}(\mathrm{M}) = M_{ij}$ for $i,j = 1 \ldots n$. The map $\det : \mathbb{R}^{n^2} \to \mathbb{R}$ is polynomial and so continuous. Therefore, as $\{0\}$ is a closed subset of $\mathbb{R}$, $\{\mathrm{M} \in M(n,\mathbb{R}) \mid \det(\mathrm{M}) = 0\}$ is a closed subset of $M(n,\mathbb{R})$. But $GL(n,\mathbb{R})$ is just the complement of this subset, so $GL(n,\mathbb{R})$ is an open subset of $M(n,\mathbb{R})$. As such $GL(n,\mathbb{R})$ is also a differentiable manifold and inherits the global coordinate system of $M(n,\mathbb{R})$. That $GL(n,\mathbb{R})$ is indeed a Lie group now follows easily since both multiplication and taking inverses are algebraic and hence differentiable. The tangent space, $T_{\mathrm{I}}(GL(n,\mathbb{R}))$ at the identity in $GL(n,\mathbb{R})$ is spanned by the n^2 partial derivatives, $(\partial/\partial x_{ij})_{\mathrm{I}}$ for $i,j = 1 \ldots n$, and is isomorphic as a vector space to $M(n,\mathbb{R})$. The Lie bracket on $T_{\mathrm{I}}(GL(n,\mathbb{R}))$ is provided by the matrix commutator, $[\mathrm{M},\mathrm{N}] = \mathrm{MN} - \mathrm{NM}$, and equipped with this bracket, $T_{\mathrm{I}}(GL(n,\mathbb{R}))$ becomes the real Lie algebra $\mathfrak{gl}(n,\mathbb{R})$. Actually, it is clear that $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$ are *real analytic* manifolds ($GL(n,\mathbb{C})$ may equally be regarded as a *complex analytic* manifold in which case it will be denoted $GL_n(\mathbb{C})$). It can be shown that the exponential map in both cases is the usual *matrix exponential map*, i.e.

$$\exp(\mathrm{M}) = e^{\mathrm{M}} = 1 + \mathrm{M} + \frac{\mathrm{M}^2}{2} + \cdots + \frac{\mathrm{M}^n}{n!} + \ldots, \tag{1.13.6}$$

where $\mathrm{M} \in \mathfrak{gl}(n,k)$ and e^{M} is the matrix exponential function and $k = \mathbb{R}$ or $\mathbb{C}$.

1.14. The Classical Lie groups

Most of the the groups which we are interested in have concrete realisations as matrix subgroups of the matrix groups $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$ of respectively real and complex invertible matrices. The groups then inherit their (Hausdorff) topology from the ambient metric spaces $\mathbb{R}^{n^2}$ or $\mathbb{C}^{n^2}$ respectively — the distance being defined by

$$\delta(\mathrm{M},\mathrm{N}) = \sqrt{\sum_{i,j=1}^{n} |M_{ij} - N_{ij}|^2} \tag{1.14.1}$$

for any two group elements M and N. In this context the otherwise difficult concepts of closure, boundedness, compactness and connectedness become more tangible. The following result is extremely useful.

THEOREM 1.14.1. *If $H \subset G$ is a subgroup of G which is closed as a topological subspace of G, then H is also a Lie group (a* Lie subgroup*). If $\mathfrak{h}$ and $\mathfrak{g}$ are the respective Lie algebras of H and G, then the differential, $d\iota|_I : \mathfrak{h} \to \mathfrak{g}$, of the inclusion map $\iota : H \to G$ is an isomorphism of $\mathfrak{h}$ with a Lie subalgebra of $\mathfrak{g}$. In fact $\mathfrak{h} = \{x \in \mathfrak{g} \mid \exp(tx) \in H for\ all\ t \in \mathbb{R}\}$.*

In particular, for any group of matrices, G, which is a closed subgroup of $GL(n,k)$, we deduce that it is a Lie subgroup of $GL(n,k)$. Its Lie algebra, $\mathfrak{g}$, is a Lie algebra of matrices with the matrix commutator providing the Lie bracket, the exponential map being the matrix exponential function and so

$$\mathfrak{g} = \{\mathrm{M} \in \mathfrak{gl}(n,k) \mid e^{t\mathrm{M}} \in G \text{ for all } t \in \mathbb{R}\}, \tag{1.14.2}$$

where $\mathrm{M} \in \mathfrak{gl}(n,k)$.

The fact that the exponential map is a diffeomorphism between open neighbourhoods of 0 in $\mathfrak{g}$ and $\mathbf{I}$ in G means that we may coordinatise G as follows. Denote by log, the inverse of the map e. Then $(U_{\mathbf{I}}, \log)$ is a chart around $\mathbf{I} \in G$. For any other point, $\mathrm{M} \in G$, L_{M} is a diffeomorphism and so $U_{\mathrm{M}} = L_{\mathrm{M}} U_{\mathbf{I}}$ is an open neighbourhood of M and defining $\phi_{\mathrm{M}} = \log \circ L_{\mathrm{M}}^{-1}$, $(U_{\mathrm{M}}, \phi_{\mathrm{M}})$ is a chart around M. It is clear that these charts fit together differentiably to provide the required differentiable manifold structure. In fact the analyticity of the matrix exponential map ensures that these groups are real analytic Lie groups.

The real classical groups will now be defined.

DEFINITION 1.14.2. $SU(l+1)$, the *special unitary group*, is the subgroup of $GL(l+1,\mathbb{C})$ $(l \geq 1)$ consisting of matrices $\mathbf{U}$ such that $\mathbf{U}\mathbf{U}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}$ and $\det(\mathbf{U}) = 1$ where $\mathbf{U}^*$ is the hermitian adjoint (i.e. $(\mathbf{U}^*)_{ij} = \overline{U_{ji}}$ where $\overline{z}$ is the complex conjugate of the complex number z).

DEFINITION 1.14.3. $SO(l)$, the *special orthogonal group*, is the subgroup of $GL(l,\mathbb{R})$ $(l \geq 3)$ consisting of matrices $\mathbf{O}$ such that $\mathbf{O}\mathbf{O}^t = \mathbf{O}^t\mathbf{O} = \mathbf{I}$ and $\det(\mathbf{O}) = 1$ where $\mathbf{O}^t$ is the transpose of $\mathbf{O}$.

DEFINITION 1.14.4. $Sp(2l)$, the *symplectic group*, is the subgroup of $GL(2l,\mathbb{C})$ $(\lambda \geq 2)$ consisting of matrices $\mathbf{U}$ such that $\mathbf{U}\mathbf{U}^* = \mathbf{U}^*\mathbf{U} = \mathbf{I}$ and $\mathbf{U}\mathbf{J}\mathbf{U}^t = \mathbf{J}$ where J was given in (1.3.15). (It follows that $\det(U) = 1$.)

It turns out to be necessary to distinguish the special orthogonal groups of the form $SO(2l+1)$ and $SO(2l)$. With the aid of Theorems 1.14.1 and 1.13.12, the following theorem may be established.

THEOREM 1.14.5. *The classical Lie groups $SU(l+1)$, $SO(2l+1)$, $Sp(2l)$ and $SO(2l)$ are compact, connected Lie groups of dimensions $l(l+2)$, $l(2l+1)$, $l(2l+1)$ and $l(2l-1)$ respectively. Their Lie algebras are denoted by $\mathfrak{su}(l+1)$, $\mathfrak{so}(2l+1)$, $\mathfrak{sp}(2l)$ and $\mathfrak{so}(2l)$ respectively. $\mathfrak{su}(l+1)$ is the Lie algebra of skew-Hermitian matrices of trace zero, $\mathfrak{so}(2l+1)$ and $\mathfrak{so}(2l)$ are Lie algebras of real skew-symmetric matrices and $\mathfrak{sp}(2l)$ is the Lie algebra of skew-Hermitian matrices,* T*, satisfying $\mathbf{T}\mathbf{J} + \mathbf{J}\mathbf{T}^t = 0$ where J was given in (1.3.15).*

1.15. The algebra of representative functions on a compact group

DEFINITION 1.15.1. A *representation* of a compact Lie group G is a pair (π, V) where $\pi : G \to GL(V)$ is a continuous group map. We say that V is a G-module with a continuous action, $\triangleright : G \times V \to V$, written as $g \triangleright v = \pi(g)v$ for all $g \in G$ and $v \in V$. This action is understood to be linear on V, but of course the notion of linearity is not present for the group.

DEFINITION 1.15.2. A *representative function* on G of a finite dimensional representation, (π, V), is a continuous function, $\pi_{\alpha,v} : G \to \mathbb{C}$ such that for any $g \in G$, $\pi_{\alpha,v}(g) = \alpha(g \triangleright v)$ where $v \in V$ and $\alpha \in V^*$. The representative functions of all finite-dimensional representations of G are denoted by $C_{\mathrm{alg}}(G)$.

As we assume V to be a finite, n-dimensional complex vector space, i.e. $V \cong \mathbb{C}^n$. Choosing a basis, $\{v_i\}_{i=1}^n$, for V results in a *matrix representation*, $\pi : G \to GL(n, \mathbb{C})$, and we denote by $\pi_{ij}(g)$ the matrix elements of the representation matrix $\pi(g)$ of a group element g such that $g \triangleright v_i = \sum_{j=1}^n \pi_{ji}(g)v_j$. The continuous, complex valued functions π_{ij} are representative functions called simply the *matrix elements* of the representation. The complex conjugate of a matrix element, $\overline{\pi_{ij}}$, is defined by $\overline{\pi_{ij}}(g) = \overline{\pi_{ij}(g)}$. The space spanned by the matrix elements of a finite dimensional representation is independent of the choice of basis and indeed the matrix elements span the space of representative functions of the representation.

The notions of submodule, reducibility, complete reducibility and irreducibilty are all defined in the obvious way. As for finite groups, every finite-dimensional representation of a compact Lie group is equivalent to one by unitary matrices. Moreover, we have the following important result concerning their reducibility.

THEOREM 1.15.3. *All finite dimensional matrix representations of a compact Lie group are completely reducible.*

Once again linear algebraic constructions for G-modules are important. Given two G-modules, V and W, with respective actions, $g \triangleright v = \pi(g)v$ and $g \triangleright w = \xi(g)w$ for all $g \in G$ and, $v \in V$ and $w \in W$, their direct sum is also a G-module with action $g \triangleright (v \oplus w) = \pi(g)v + \xi(g)w$ and so is their tensor product, with action $g \triangleright (v \otimes w) = (\pi(g)v) \otimes (\xi(g)w)$ for all $v \in V$ and $w \in W$. Any constant multiple, cV of a G-module is again a G-module, and so is the dual vector space, V^*, with action, $(g \triangleright \alpha)(v) = \alpha(g^{-1} \triangleright v)$ for all $\alpha \in V^*$. Notice that in terms of matrix elements, if $\{\alpha_i\}_{i=1}^n$ and $\{v_i\}_{i=1}^n$ are dual bases of V^* and V and if $g \triangleright \alpha_i = \sum_{j=1}^n \xi_{ji}(g)\alpha_j$ and $g \triangleright v_i = \sum_{j=1}^n \pi_{ji}(g)v_j$ then $\xi_{ij}(g) = \pi_{ji}(g^{-1})$. Related to the dual representation is the conjugate representation. This is obtained by introducing the space $\overline{V}$ defined as the same underlying additive group as V but with the scalar multiplication defined as $c \triangleright v = \bar{c}v$ for any $c \in \mathbb{C}$. If V is a G-module then so is $\overline{V}$ with $g \triangleright v_i = \sum_{j=1}^n \overline{\pi_{ji}(g)}v_j$. Since all finite dimensional representations are equivalent to representations in terms of unitary matrices we can see that V and $\overline{V}$ are equivalent as G-modules.

The natural unit for the algebra of continuous functions on G is the identity function, $\mathbf{1} : G \to \mathbb{R}$, such that $\mathbf{1}(g) = 1$ for all $g \in G$. It is also a representative function since on any vector space, V, we can define the trivial representation where each group element acts as the identity, i.e. $g \triangleright v = v = \mathbf{1}(g)v$ for all $v \in V$.

The space of representative functions, $C_{\mathrm{alg}}(G)$, which by the previous theorem is spanned by the representative functions of the irreducible representations, forms a subalgebra of the algebra of all continuous complex-valued functions on G, $C(G)$. In fact $C_{\mathrm{alg}}(G)$ has a Hopf algebra structure which is formally precisely that in Theorem 1.7.5, but with ρ replaced with π, and $\eta(1) = \mathbf{1}$. There is actually some further structure on $C_{\mathrm{alg}}(G)$ for which we need the following definition.

DEFINITION 1.15.4. A $\star$-*algebra* is an algebra over $\mathbb{C}$, (A, m, η) say, endowed with a conjugate linear map $\star : A \to A$ such that

$$\star \circ \star = \mathrm{id}, \quad m \circ (\star \otimes \star) \circ P = \star \circ m, \tag{1.15.1}$$

where $P(a \otimes a') = a' \otimes a$ for all $a, a' \in A$. A $\star$-algebra map, $\phi : A \to A'$, between two $\star$-algebras, A and A', is an algebra map which satisfies $\phi \circ \star = \star \circ \phi$. Dually, a $\star$-*coalgebra* is a coalgebra over $\mathbb{C}$, (C, Δ, η) say, equipped with a linear map $\bullet : C \to C$ such that

$$\bullet \circ \bullet = \mathrm{id}, \quad P \circ (\bullet \otimes \bullet) \circ \Delta = \Delta \circ \bullet. \tag{1.15.2}$$

A *Hopf* $\star$-*algebra* is a Hopf algebra A such that the constituent algebra is a $\star$-algebra with the coalgebra maps, Δ and ϵ, $\star$-algebra maps. The constituent coalgebra is then a $\star$-coalgebra by the map $\star \circ S$.

It is usual to denote the $\star$-map on elements a of a given Hopf $\star$-algebra as $a^\star$. Then on matrix elements π_{ij} of some representation (π, V) of G, we define $\pi^\star_{ij}(g) = \overline{\pi_{ij}(g)}$ or, denoting the matrix coefficients of the conjugate representation $\overline{V}$ by $\overline{\pi}_{ij}$, $\pi^\star_{ij} = \overline{\pi}_{ij}$.

The representation theory of $C_{\mathrm{alg}}(G)$ is of course not particularly interesting as it is a commutative Hopf algebra. However the corepresentation theory corresponds to the representation theory of G and so is extremely interesting. Indeed, given a finite dimensional continuous representation (π, V) of G, we define a corepresentation $(\Delta_V, C_{\mathrm{alg}}(G))$ by $\Delta_V(v)(g) = \pi(G)$ where we are making use of the isomorphism $V \otimes C(G) \cong C(G, V)$ where by $C(G, V)$ we mean the continuous vector-valued functions on G. In the other direction, it is clear that starting with a corepresentation of $C_{\mathrm{alg}}(G)$ we recover a continuous representation of G. This correspondence preserves the usual representation attributes (i.e. irreducibility etc.). Moreover, there is a natural definition of a unitary corepresentation and with this unitary representations correspond to unitary corepresentations.

1.16. The Stone-Weierstrass theorem and the algebra of representative functions on the classical Lie groups

The definitions of the classical Lie groups provide 'tautological' faithful matrix representations which we shall denote by (r, V) (incidentally, these representations are irreducible). In each case the group space is a metric space which is moreover compact so $C(G)$ may be equipped with the *supremum norm topology* provided by the norm $|f| = \sup\{|f(g)| \mid \text{for all } g \in G\}$. The matrix coefficients of the respective defining representations 'separate points' of G in the sense that if two group elements are not the same then there must be a matrix coefficient which distinguishes them. The following theorem of analysis, called the *Stone-Weierstrass theorem,* is of fundamental importance.

THEOREM 1.16.1. *Let E be a compact metric space. If a subalgebra A of the algebra of complex valued functions on E, $C(E)$, contains all complex constants, separates points* *of E and is such that for each $f \in A$ the conjugate function $\overline{f}$ also belongs to A then A is dense in $C(E)$.*

Now, for any of the classical Lie groups we can form the polynomial algebra over $\mathbb{C}$, denoted $\mathbb{C}[G]$, which is generated by the matrix elements r_{ij} and their complex conjugates, $\overline{r_{ij}}$. Of course the r_{ij} and $\overline{r_{ij}}$ belong to $C_{\mathrm{alg}}(G)$ and moreover by the Stone-Weierstrass theorem $\mathbb{C}[G]$ is dense in $C(G)$ so we deduce that $C_{\mathrm{alg}}(G)$ is dense in $C(G)$. In the more general context of an arbitrary compact Lie group this result is the celebrated *Peter-Weyl theorem.* With a little further work, the relationship between $\mathbb{C}[G]$ and $C_{\mathrm{alg}}(G)$ may be clarified.

THEOREM 1.16.2. *The algebras $\mathbb{C}[G]$ and $C_{\mathrm{alg}}(G)$ are isomorphic.*

In the special cases of the classical Lie groups we can actually make a further refinement. In these cases it turns out that $\mathbb{C}[G]$ is generated already by the r_{ij}. Denoting by $P(r(g))$, polynomials in the matrix elements of (r, V) and by $\mathbf{I}_n$, the $n \times n$ identity matrix, the details in each case are as follows:

1. $SU(n)$

   The first fundamental representation of $SU(n)$ is precisely the (faithful) defining representation in terms of unitary matrices with determinant 1. Thus for all $g \in SU(n)$, each matrix, $r(g)$, in this representation satisfies $r(g)r(g)^* = r(g)^*r(g) = \mathbf{I}_n$ so $r(g)^* = r(g)^{-1}$. But $(r(g)^{-1})_{ij} = P(r(g))/\det(r(g))$ and $\det(r(g)) = 1$ so that $\overline{r(g)_{ij}} = P(r(g))$. That is, the complex conjugate matrix elements, $\overline{r_{ij}}$, are polynomials in the matrix elements r_{ij}.
2. $SO(n)$

   Again, the first fundamental representation of $SO(n)$ is just the (faithful) defining representation, now in terms of *real* orthogonal matrices. Thus, in this case it is immediate that for all $g \in G$, $\overline{r_{ij}(g)} = r_{ij}(g)$, so $\overline{r_{ij}} = r_{ij}$.
3. $Sp(2n)$

   Once again, the first fundamental representation of $Sp(2n)$ is just the (faithful) defining representation in terms of $2n \times 2n$ matrices, $r(g)$, satisfying, in particular, $r(g)r(g)^* = r(g)^*r(g) = \mathbf{I}_{2n}$ and $\det(r(g)) = 1$ for all $g \in Sp(2n)$. By the same reasoning as for $SU(n)$ we conclude that the complex conjugate matrix elements, $\overline{r_{ij}}$, are polynomials in the matrix elements π_{ij}.

Since the matrix representations (r, V), and in particular their matrix elements, are certainly analytic, it now follows that the matrix elements of *all* finite-dimensional representations are analytic so indeed the representations themselves are analytic. Also, it is clear that we can view the Hopf algebras $\mathbb{C}[G]$ as being generated by the matrix elements of their first fundamental representations of G.

EXAMPLE 1.16.3. Let us consider $\mathbb{C}[SU(2)]$. This commutative Hopf algebra is generated by the matrix elements of the first fundamental representation, r_{11}, r_{12}, r_{21} and r_{22} say. We will denote the unit of $\mathbb{C}[SU(2)]$ by $\mathbf{1}$. The coalgebra structure is just $\Delta(\pi_{ik}) = \sum_{j=1}^{2} \pi_{ij} \otimes \pi_{jk}$ and $\epsilon(\pi_{ij}) = \delta_{ij}$. The fact that $\det(\mathbf{U}) = 1$ for all $\mathbf{U} \in SU(2)$ leads immediately to the corresponding relation

$$r_{11}r_{22} - r_{12}r_{21} = \mathbf{1} \tag{1.16.1}$$

for the generators of $\mathbb{C}[SU(2)]$. In fact we write such a relation also as $\det(r) = \mathbf{1}$. The antipode and $\star$-structure on $\mathbb{C}[SU(2)]$ are similarly obtained immediately by considering the matrix elements of $\mathbf{U}^{-1}$ for any $\mathbf{U} \in SU(2)$

$$S\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \begin{pmatrix} r_{22} & -r_{12} \\ -r_{21} & r_{11} \end{pmatrix} = \begin{pmatrix} r_{11}^\star & r_{21}^\star \\ r_{12}^\star & r_{22}^\star \end{pmatrix}. \tag{1.16.2}$$

Compare this Hopf algebra (without $\star$-structure) with $U(\mathfrak{sl}_2(\mathbb{C}))^{(\circ)}$ obtained in 1.12.6.

For the other classical Lie groups the Hopf algebra structure of $\mathbb{C}[G]$ is just as straightforward to obtain. In each case we have the determinant condition $\det(r) = \mathbf{1}$ and for the orthogonal groups also

$$rr^t = \mathbf{I} \tag{1.16.3}$$

while for the symplectic groups

$$\rho^t \mathbf{J} \rho \mathbf{J}^{-1} = \mathbf{I}. \tag{1.16.4}$$

Furthermore, the antipode is given by $S(r_{ij}) = r^{ji}$ where r^{ij} is the 'cofactor' of the element r_{ij} in the 'matrix' r. Finally, the $\star$-structure is given on the generators by $r_{ij}^\star = r^{ij}$ and extended conjugate linearly and homomorphically to the whole of $\mathbb{C}[G]$.

1.17. The compact real forms of the classical complex Lie algebras and their representations

The complex simple Lie algebras, $\mathfrak{sl}_{l+1}(\mathbb{C})$, $\mathfrak{so}_{2l+1}(\mathbb{C})$, $\mathfrak{sp}_{2l}(\mathbb{C})$ and $\mathfrak{so}_{2l}(\mathbb{C})$ are respectively complexifications of the real simple Lie algebras $\mathfrak{su}(l+1)$, $\mathfrak{so}(2l+1)$, $\mathfrak{sp}(2l)$ and $\mathfrak{so}(2l)$ according to the following definition:

DEFINITION 1.17.1. A real Lie algebra $\mathfrak{g}_\mathbb{R}$ is a *real form* of a complex Lie algebra $\mathfrak{g}$ if $\mathfrak{g} \cong \mathbb{C} \otimes_\mathbb{R} \mathfrak{g}_\mathbb{R}$. In this case $\mathfrak{g}$ is called the *complexification* of $\mathfrak{g}_\mathbb{R}$.

A remarkable theorem due to Weyl says that *a matrix Lie group G is compact if and only if the Killing form of its Lie algebra is negative definite.* In this case the Lie algebra, $\mathfrak{g}$, is said to be *compact.*

Every complex simple Lie algebra has a real form which is also compact. This is made explicit in the following theorem:

THEOREM 1.17.2. *In terms of the Cartan-Weyl basis of a complex simple Lie algebra, $\mathfrak{g}$, the elements iH_i, $E_\alpha + E_{-\alpha}$ and $i(E_\alpha - E_{-\alpha})$ form the basis of real simple Lie algebra which is denoted, $\mathfrak{g}_{\text{cpt}}$, and called the compact real form of $\mathfrak{g}$.*

The irreducible representations of the four series of classical complex Lie algebras, A_l, B_l, C_l and D_l, restrict as irreducible representations to the respective compact real forms, $\mathfrak{su}(l+1)$, $\mathfrak{so}(2l+1)$, $\mathfrak{sp}(2l)$ and $\mathfrak{so}(2l)$ respectively. Moreover it is a general result that *all irreducible representations of simple real Lie algebras whose complexifications are simple arise as the restrictions of the irreducible representations of the complex Lie algebras.*

It is worth mentioning that we assume our $\mathfrak{g}$-modules to be generic complex vector spaces. However when we regard them as representations of the compact real forms, $\mathfrak{g}_{\text{cpt}}$, it may be that some are equivalent to real representations. This is true in the particular cases of the first fundamental representations of B_l and D_l — as we would expect!

1.18. Representations of the Classical Lie groups

To every Lie group G there corresponds a unique Lie algebra $\mathfrak{g}$. In the other direction, given $\mathfrak{g}$ there may correspond more than one Lie group. However, amongst the connected Lie groups sharing $\mathfrak{g}$ there is a distinguished one called the *universal covering group* and denoted $\tilde{G}$. This group is simply-connected, unique up to isomorphism and such that every other connected group G whose Lie algebra is $\mathfrak{g}$ is given by $G \cong \tilde{G}/\pi_1(G)$ where $\pi_1(G)$ is the fundamental group of G.

The compact connected groups $SU(l+1)$ and $Sp(2l)$ are simply-connected while $SO(2l+1)$ and $SO(2l)$ are not and have universal covering groups $Spin(2l+1)$ and $Spin(2l)$ respectively, with

$$SO(n) \cong Spin(n)/\mathbb{Z}_2. \tag{1.18.1}$$

If (π_G, V) is a continuous representation of some matrix Lie group, G, in terms of matrices, $\pi_G(g)$, for all $g \in G$, then (π_L, V) is a representation of the corresponding Lie algebra, $\mathfrak{g}$, in terms of matrices

$$\pi_L(\mathbf{x}) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \pi_G(e^{t\mathbf{x}}) \tag{1.18.2}$$

for all $\mathbf{x} \in \mathfrak{g}$. If (π_L, V) is obtained from (π_G, V) in this way, we say that the Lie algebra representation *exponentiates* to a Lie group representation. As the classical groups are connected, equivalence, irreducibility and complete reducibility are all preserved through both 'differentiation' and exponentiation.

The fundamental theorem is the following:

THEOREM 1.18.1. *If G_1 and G_2 are Lie groups (real or complex complex analytic) with G_1 simply connected and $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are the corresponding Lie algebras, then a linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is the differential $\mathrm{d}\psi$ of a Lie group homomorphism $\psi : G_1 \to G_2$ if and only if ϕ is a Lie algebra homomorphism.*

This result tells us that in the cases of $SU(n)$ and $Sp(n)$, not only is there a Lie algebra representation (π_L, V) corresponding to each group representation, (π_G, V), but that *every* Lie algebra representation arises in this way. In other words, in the cases $SU(l+1)$ and $Sp(2l)$, *every Lie algebra representation exponentiates to a Lie group representation.*

This is not the case for the Lie groups $SO(2l+1)$ and $SO(2l)$ as they are not simply-connected. In these cases not all Lie algebra representations exponentiate. That is, there are Lie algebra representations which are not obtained from group representations by differentiating. It is true however that *all* Lie algebra representations exponentiate to representations of the respective universal covering groups, $Spin(2n+1)$ and $Spin(2n)$. It is then a question of which of these representations factor through the quotient $\mathbb{Z}_2$. This turns out to be reasonably straightforward to establish. The result is as follows:

THEOREM 1.18.2.
- *$SO(2l+1)$: Of the irreducible representations of $\mathfrak{so}(2l+1)$, with highest weight, $\Lambda = n_1\Lambda_{(1)} + n_2\Lambda_{(2)} + \ldots + n_l\Lambda_{(l)}$, those that exponentiate to representations of $SO(2l+1)$ are such that n_l* is even.
- *$SO(2l)$: Of the irreducible representations of $\mathfrak{so}(2l)$, with highest weight, $\Lambda = n_1\Lambda_{(1)} + n_2\Lambda_{(2)} + \ldots + n_l\Lambda_{(l)}$, those that exponentiate to representations of $SO(2l)$ are such that $n_{l-1} + n_l$* is even.

We see that the irreducible representations of the classical Lie algebras, $\mathfrak{su}(l+1)$, $\mathfrak{so}(2l+1)$, $\mathfrak{sp}(2l)$ and $\mathfrak{so}(2l)$, which exponentiate to representations of the corresponding classical Lie groups $SU(l+1)$, $SO(2l+1)$, $Sp(2l)$ and $SO(2l)$ *are precisely those that can be obtained in tensor products of the respective defining representations.* That each irreducible representation of these *groups* appears in tensor products of the respective defining representations is then an immediate consequence of this *Lie algebraic* result!

In each of the cases dealt with in the previous theorem, the weights of the representations which exponentiate form additive subgroups, $\Lambda(G)$ where $G = SO(2l+1), SO(2l)$, of the respective weight lattices P of the Lie algebras B_l and D_l. That is, in each case we have $Q \subset \Lambda(G) \subset P$. Remarkably, $\pi_1(G) \cong P/\Lambda(G)$ and $Z(G) \cong \Lambda(G)/Q$, results which have precise generalisations.

1.19. Complexification of the real classical groups

For each of the classical compact Lie groups $G = SU(l+1)$, $SO(2l+1)$, $Sp(2l)$ and $SO(2l)$, the algebra of representative functions, $\mathbb{C}[G]$, carries the structure of a Hopf $\star$-algebra. Through Tanaka-Krein duality, the respective compact groups may be reconstructed from these Hopf algebras as the set, $\mathrm{Hom}_{\star\text{-algebra}}(\mathbb{C}[G], \mathbb{C})$, of $\star$-algebra maps from $\mathbb{C}[G]$ to $\mathbb{C}$ (endowed with the natural group structure coming from the Hopf structure of $\mathbb{C}[G]$ and the coarsest topology such that the evaluation maps $\lambda_s : \mathrm{Hom}_{\star\text{-algebra}}(\mathbb{C}[G], \mathbb{C}) \to \mathbb{C}$ given by $\lambda_s(\phi) = \phi(s)$ are all continuous).

If we forget about the $\star$-structure and denote by $G^{\mathbb{C}}$ the set $\mathrm{Hom}_{\mathbb{C}\text{-algebra}}(\mathbb{C}[G], \mathbb{C})$, then $G^{\mathbb{C}}$ is precisely the *affine algebraic variety* specified by the *coordinate ring* $\mathbb{C}[G]$ (by the coordinate ring $\mathbb{C}[G]$ we will understand $\mathbb{C}[G]$ without $\star$-structure). Indeed, we saw in Section 1.16 that for each of the classical compact Lie groups, disregarding the $\star$-structure, $\mathbb{C}[G]$ has the form $\mathbb{C}[r_{ij}]/I$ where $\mathbb{C}[r_{ij}]$ is the polynomial algebra over $\mathbb{C}$ in the symbols r_{ij}, $i, j = 1 \dots n$, and I is a certain ideal (e.g. in the case of $SU(l+1)$ this is just the ideal generated by the expression $\det(r) - 1$). It is then standard that $\mathrm{Hom}_{\mathbb{C}\text{-algebra}}(\mathbb{C}[G], \mathbb{C})$ is in bijection with the set of common zeros of the polynomials in I, that is, the affine algebraic variety $V(I)$. Furthermore, recognising $GL_n(\mathbb{C})$ as a complex linear algebraic group, the $V(I)$ are clearly subgroups which are moreover specified by polynomial expressions as subsets of $GL_n(\mathbb{C})$ and as such are *Zariski-closed* subgroups and hence complex linear algebraic subgroups. These then are the *complexifications* of the classical compact Lie groups. Explicitly, the complexifications of $SU(l+1)$, $SO(2l+1)$, $Sp(2l)$ and $SO(2l)$ are $SL_{l+1}(\mathbb{C})$, $SO_{2l+1}(\mathbb{C})$, $Sp_{2l}(\mathbb{C})$ and $SO_{2l}(\mathbb{C})$ respectively. They will be called the complex classical Lie groups. They are again matrix groups, which as complexifications can be regarded either as complex linear algebraic groups or as complex analytic groups. Their Lie algebras are precisely the complex simple Lie algebras, $\mathfrak{sl}_{l+1}(\mathbb{C})$, $\mathfrak{so}_{2l+1}(\mathbb{C})$, $\mathfrak{sp}_{2l}(\mathbb{C})$ and $\mathfrak{so}_{2l}(\mathbb{C})$ respectively, that is, the complexifications of the Lie algebras of the compact Lie groups.

The coordinate rings $\mathbb{C}[G]$ are precisely the Hopf algebras generated by the matrix elements of the defining representations of the respective complexified groups $G^{\mathbb{C}}$. Furthermore, there is a one-to-one correspondence between corepresentations of $\mathbb{C}[G]$ and rational representations of $G^{\mathbb{C}}$ (i.e. those representations whose matrix elements are polynomial functions on $G^{\mathbb{C}}$).

Regarding the complex classical groups as complex analytic Lie groups, they are connected, with $SL_{l+1}(\mathbb{C})$ and $Sp_{2l}(\mathbb{C})$ simply connected. The complex orthogonal groups are not simply connected, and we have $SO_n(\mathbb{C}) \cong Spin_n(\mathbb{C})/\mathbb{Z}_2$ where $Spin_n(\mathbb{C})$, the universal covering group of $SO_n(\mathbb{C})$, is the complexification of $Spin(n)$. Each finite dimensional representation of $\mathfrak{sl}_{l+1}(\mathbb{C})$ and $\mathfrak{sp}_{2l}(\mathbb{C})$ exponentiates to a complex analytic representation of the respective groups, $SL_{l+1}(\mathbb{C})$ and $Sp_{2l}(\mathbb{C})$. The representations of $\mathfrak{so}_{2l+1}(\mathbb{C})$ and $\mathfrak{so}_{2l}(\mathbb{C})$ which exponentiate respectively to complex analytic representations of $SO_{2l+1}(\mathbb{C})$ and $SO_{2l}(\mathbb{C})$ are precisely those whose restrictions to the compact real forms exponentiated to representations of the corresponding compact groups. The defining representations of the complex classical Lie groups (which are irreducible) appear in their very construction and the restriction of these representations to the compact Lie groups within them are precisely the defining representations of the classical compact Lie groups.

In the case of the the real classical Lie groups and their real Lie algebras we were able to deduce, from the tensor structure of the Lie algebra representations together with a knowledge of which representations exponentiated, that every irreducible representation of the classical compact Lie groups appear in tensor products of the respective defining representations. Precisely the same reasoning tells us that for all the classical complex Lie groups, each irreducible complex analytic representation may be obtained within tensor products of the respective defining representations. It follows that the matrix elements of the finite dimensional representations of the classical complex Lie groups are polynomials in the matrix elements of the group elements. This in turn means that every finite complex analytic representation of a classical complex Lie group is a rational representation in the sense of complex linear algebraic groups. Thus we see that there is a one-to-one correspondence between corepresentations of $\mathbb{C}[G]$ and complex analytic representations of $G^{\mathbb{C}}$.

1.20. $U(\mathfrak{g})^{(\circ)}$ and $\mathbb{C}[G]$ are isomorphic

Suppose G (we are dropping the superscript $\mathbb{C}$) is the complexification of one of the classical compact Lie groups and $\mathfrak{g}$ is its Lie algebra.

THEOREM 1.20.1. *There is a non-degenerate Hopf algebra pairing between the Hopf algebras $U(\mathfrak{g})$ and $\mathbb{C}[G]$. It is given by the action of elements of $U(\mathfrak{g})$ on $\mathbb{C}[G]$ as left-invariant differential operators at the identity, explicitly*

$$\langle P, X_1X_2\dots X_k\rangle = \left.\frac{\partial^k}{\partial t_1\partial t_2\dots\partial t_k}\right|_{t_1=t_2=\dots t_k=0} P(e^{t_1X_1}e^{t_2X_2}\dots e^{t_kX_k}), \qquad (1.20.1)$$

where $P \in \mathbb{C}[G]$ and $X_i \in \mathfrak{g}$.

As a consequence of this important theorem, there is an embedding $\iota : \mathbb{C}[G] \to U(\mathfrak{g})^*$, of $\mathbb{C}[G]$ in $U(\mathfrak{g})^*$ as a sub-Hopf algebra. We have already seen that complex analytic representations of G are in one-to-one correspondence with corepresentations of $\mathbb{C}[G]$ and in fact that $\mathbb{C}[G]$ is generated by the matrix elements r_{ij}, of the defining representation of G, (r, V). From Section 1.9 we deduce that the $\iota(r_{ij})$ may be identified with the matrix elements ρ^V_{ij} of the defining representation of $\mathfrak{g}$. It is then a question of which $U(\mathfrak{g})$-modules exponentiate to corresponding G-modules. We have already found the answer to this in Section 1.18 and so can deduce the important isomorphism

$$\mathbb{C}[G] \cong U(\mathfrak{g})^{(\circ)}. \qquad (1.20.2)$$

CHAPTER 2

The quantum picture

2.1. Introduction

The purpose of this chapter is to provide the reader with an overview, in the form of key definitions and results, of the structural elements of quantum group theory. Again we restrict our discussion to structures associated with complex simple Lie algebras and in places the classical complex simple Lie algebras.

We begin with motivation and discussion of the concepts of h-adic topology and completion. With this mathematical machinery we are then able to formulate in a mathematically rigorous manner the basic definitions and results concerning topological quasitriangular Hopf algebras, quantised universal enveloping algebras and their representations.

The quantised enveloping algebras are deformations of classical enveloping algebras. We recall two key cohomology results. The first tells us that there are no non-trivial deformations of universal enveloping algebras as algebras, but they may have non-trivial Hopf algebra deformations. The second tells us that there are no non-trivial deformations of representations of universal enveloping algebras. One consequence of these results is that there is a bijective correspondence between representations of the classical $U(\mathfrak{g})$ and the quantum $U_h(\mathfrak{g})$ such that, for example, Clebsch-Gordan series are preserved. Already these results suggest that there must be a quasitriangular quasi-Hopf algebra associated with $U_h(\mathfrak{g})$ and this is confirmed by a third cohomolgy result.

The least ad-hoc construction of the quantised version of the coordinate rings of classical Lie groups is to define these through the appropriate Hopf dual of the quantised universal enveloping of the corresponding Lie algebra. The classical theory described in Chapter 1 is the motivating model here. There is also a construction due originally to Faddeev, Reshetikhin and Takhtajan (FRT) and later developed by Majid. The FRT construction requires as input only a numerical R-matrix and always provides a bialgebra which can loosely be regarded as some sort of deformation of the classical bialgebra of regular functions on matrices. When the numerical R-matrix is the one obtained in the first fundamental representation from the universal R-matrix of one of the Drinfeld-Jimbo quantised enveloping algebras there is an obvious similarity between the FRT quantum group quantisations of the classical groups $SL_{l+1}(\mathbb{C})$, $SO_{2n+1}(\mathbb{C})$, $Sp_{2l}(\mathbb{C})$ and $SO_{2l}(\mathbb{C})$ and the quantisations defined as the appropriate Hopf dual of the objects $U_h(\mathfrak{sl}_{l+1}(\mathbb{C}))$, $U_h(\mathfrak{so}_{2l+1}(\mathbb{C}))$, $U_h(\mathfrak{sp}_{2l}(\mathbb{C}))$ and $U_h(\mathfrak{so}_{2l}(\mathbb{C}))$ respectively. Quantum group folk-lore says that these objects are the same though a proof of this in any of the cases other than $SL_{l+1}(\mathbb{C})$ does not seem to have been published.

In a style somewhat analogous to considering the semiclassical Poisson structures of a quantum mechanical system, we investigate the structures which exist within quantum groups at order h in the deformation indeterminate. At this level the relevant objects are

Lie bialgebras, for which there is a complete classification. The classification of Lie bialgebras raises natural questions of existence and uniqueness of corresponding quantisations. The correct mathematical framework for the investigation of such issues is provided by Drinfeld's notions of quasi-Hopf algebras and twisting. We recall Drinfeld's famous theorem. This result together with some more recent work of Etingof and Kazhdan provide the motivation for Chapter 3.

The Jordanian quantum group will be investigated in some detail in Chapter 4. In Section 2.10 we use it to exemplify some of the discussion of the chapter.

The references for this section are first and foremost the four excellent books on quantum groups by Chari and Pressley [**17**], Kassel [**60**], Jantzen [**54**] and in particular Guichardet [**46**]. Drinfeld's original papers of course provide inspiration. h-adic topology is discussed in some detail in [**6**]. The definition of quantised coordinate rings for the groups $SO_n(\mathbb{C})$ in terms of representations of quantised enveloping algebras is difficult to find in the literature. It seems to be well known amongst researchers but I was only able to find it explicitly stated in the work of Hodges [**51**] and less explicitly in the thesis of Dijkhuizen [**28**].

2.2. h-adic topology

The quintessential example of a quantum group, $U_h(\mathfrak{sl}_2(\mathbb{C}))$, first appeared as an algebra in the work of Kulish and Reshetikhin [**64**]. Its full Hopf algebra structure was worked out later by Skylanin in [**93**]. In [**64**] a 'deformation' of $U(\mathfrak{sl}_2(\mathbb{C}))$ appeared as the algebra with generators $\{X, Y, H\}$ and commutator relations

$$\begin{aligned} [H, X] = 2X, \quad [H, Y] = -2Y, \\ [X, Y] = \frac{\sinh hH/2}{\sinh h/2}. \end{aligned} \tag{2.2.1}$$

A question immediately arises as to the interpretation of the right hand side in (2.2.1). In particular how should we understand an object such as $e^{hH/2}$ where a is an element of some algebra A. Splitting it up into partial, finite sums, $s_n = \sum_{i=1\ldots n} \frac{h^i}{i!2^i} H^i$, each s_n is well defined. What we need is a topological setting in which there is an equivalent of the usual Cauchy criterion of analysis, and in which $e^{hH} = \lim_{n\to\infty} s_n$.

DEFINITION 2.2.1. The *ring of formal power series*, $\mathbb{C}[[h]]$, is the ring (which may also be regarded as an algebra over $\mathbb{C}$) consisting of all *formal* objects of the form $\sum_{i=0\ldots\infty} a_i h^i$. If $\tilde{a}, \tilde{b} \in \mathbb{C}[[h]]$ such that $\tilde{a} = \sum_{n\geq 0} a_n h^n$ and $\tilde{b} = \sum_{n\geq 0} b_n h^n$ then

$$\tilde{a} + \tilde{b} = \sum_{n\geq 0} (a_n + b_n) h^n, \quad \tilde{a}\tilde{b} = \sum_{n\geq 0} (\sum_{p+q=n} a_p b_q) h^n. \tag{2.2.2}$$

$\mathbb{C}[[h]]$ can be understood in topological terms such that *formal* objects like $\tilde{a}$ are interpreted as *limits* of the corresponding partial sums. We consider the truncations $\mathbb{C}[[h]]/h^n\mathbb{C}[[h]]$. Denoting by S_n the natural projection $S_n : \mathbb{C}[[h]] \to \mathbb{C}[[h]]/h^n\mathbb{C}[[h]]$ we write the image of $\tilde{a}$ under S_n as $\tilde{a} \bmod h^n$. Note that $\tilde{a} \in \mathbb{C}[[h]]$ is invertible *if and only if* $\tilde{a} \bmod h \neq 0$. The algebras $\mathbb{C}[[h]]/h^n\mathbb{C}[[h]]$, for all $n \geq 1$ of polynomials in h of order up to $n-1$ can be linked chain-like by the natural projections $p_n : \mathbb{C}[[h]]/h^n\mathbb{C}[[h]] \to \mathbb{C}[[h]]/h^{n-1}\mathbb{C}[[h]]$, defined as $p_n(\sum_{i=0\ldots n-1} a_i h^i) = \sum_{i=0\ldots n-2} a_i h^i$.

The family $(\mathbb{C}[[h]]/h^n\mathbb{C}[[h]], p_n)$ is an example of an *inverse system of algebras*, that is, a family (A_n, p_n) for all $n \geq 0$ or $n \geq 1$ of algebras A_n together with maps $p_n : A_n \to A_{n-1}$.

DEFINITION 2.2.2. The *inverse limit*, $A = \varprojlim A_n$, of an inverse system of algebras is the set

$$\varprojlim A_n = \{(a_n) \in \prod_{n\geq 0} A_n \mid p_n(a_n) = a_{n-1}\}, \tag{2.2.3}$$

endowed with the usual componentwise addition and multiplication inherited from $\prod_{n\geq 0} A_n$. We define the map $\pi_k : \varprojlim A_n \to A_k$ to be the restriction to $\varprojlim A_n$ of the natural projection from $\prod_{n\geq 0} A_n$ to A_k, then $p_n \circ \pi_n = \pi_{n-1}$.

A useful and easily derived result on inverse limits is the following:

THEOREM 2.2.3. *Given an algebra B, an inverse system of algebras (A_n, p_n) and a family of algebra maps $(f_n : B \to A_n)$ such that $p_n \circ f_n = f_{n-1}$ for all $n > 0$, there exists an algebra map*

$$f : B \to \varprojlim A_n \tag{2.2.4}$$

such that $\pi_n \circ f = f_n$ for all $n \geq 0$.

Using this result it is not difficult to check that $\mathbb{C}[[h]] \cong \varprojlim \mathbb{C}[[h]]/h^n\mathbb{C}[[h]]$. Also, we find that the quotient of an inverse limit is again an inverse limit.

THEOREM 2.2.4. *If (A_n, p_n) is an inverse system of algebras with $A = \varprojlim A_n$ and I is a two-sided ideal in A then*

$$A/I \cong \varprojlim (A_n/I_n), \tag{2.2.5}$$

where $I_n = \pi_n(I)$ and π_n is the natural algebra map $\pi_n : A \to A_n$.

The topology with which we promised to endow $\mathbb{C}[[h]]$ is the so-called *inverse limit topology*. It is obtained generally, for any inverse limit $\varprojlim A_n$, by giving each algebra A_n the discrete topology, $\prod_{n\geq 0} A_n$ the induced product topology and then restricting to $\varprojlim A_n$. A basis of open sets of $\varprojlim A_n$ is then provided by the sets $\{\pi_n^{-1}(U_n)\}$ where U_n is an open set of A_n. In the particular case of $\mathbb{C}[[h]]$, open neighbourhoods of 0 in $\mathbb{C}[[h]]$ are the sets $h^n\mathbb{C}[[h]]$ for all $n > 0$ and the topology is translation invariant in that $\pi_n^{-1}(a) = a + h^n\mathbb{C}[[h]]$ where a is any element of $\mathbb{C}[h]/(h^n)$. This topology is also called the *h-adic topology.*

Actually, the h-adic topology is a metric topology. For any $\tilde{a} \in \mathbb{C}[[h]]$ define $v_h(\tilde{a})$ to be the largest nonnegative integer n such that $\tilde{a} \in h^n\mathbb{C}[[h]]$.

THEOREM 2.2.5. *$\mathbb{C}[[h]]$ is a metric space with a distance $d(\tilde{a}, \tilde{b})$ defined for any pair of elements $\tilde{a}, \tilde{b} \in \mathbb{C}[[h]]$ as $d(\tilde{a}, \tilde{b}) = e^{-v_h(\tilde{a}-\tilde{b})}$.*

DEFINITION 2.2.6. A sequence $\{s_n\}$ in $\mathbb{C}[[h]]$ is called a *Cauchy sequence* if given some P there exists an N such that for all $m, n \geq N$, $s_m - s_n \in h^P\mathbb{C}[[h]]$.

In particular, the partial sums $s_n = \sum_{i=0\dots n} a_i h^i$ form a Cauchy sequence, and we can see that they converge in $\mathbb{C}[[h]]$ to $\tilde{a} = \sum_{i=0\dots\infty} a_i h^i$, so giving 'meaning' to the infinite sum. Moreover it is not difficult to check that *every* Cauchy sequence converges in $\mathbb{C}[[h]]$ so that $\mathbb{C}[[h]]$ is *complete* in this h-adic topology.

More generally, for any $\mathbb{C}[[h]]$-module M we can construct the inverse limit of the $\mathbb{C}[[h]]$-modules M/h^nM to obtain the *completion*, $\hat{M}$, of M in the h-adic topology. In $\hat{M}$, the open neighbourhoods of 0 are the sets $h^n\hat{M}$, and we may consider elements of the form $\sum_{i=0\ldots\infty} m_i h^i$ where $m_i \in M$. $\hat{M}$ is a metric space with $d(m, m') = e^{-v_h(m-m')}$ where now $v_h(m)$ is the largest non-negative integer n such that $m \in h^n\hat{M}$. It is important to note that a completed $\mathbb{C}[[h]]$-module is both separated (Hausdorff) and complete.

In particular for algebras A over $\mathbb{C}[[h]]$ objects like e^{ha} with $a \in A$ are well defined in the h-adic completion of A, $\hat{A}$. Notice also that e^{ha} is invertible in this context as $e^{ha} \bmod h = 1$.

If a and b are elements of some algebra A over $\mathbb{C}[[h]]$, then $e^{ha\otimes b}$ is not well defined in $A \otimes_{\mathbb{C}[[h]]} A$. Generally if M and N are $\mathbb{C}[[h]]$-modules we need the h-adic completion, $M\hat{\otimes}N$, of $M \otimes_{\mathbb{C}[[h]]} N$, obtained as the inverse limit of the modules $(M \otimes_{\mathbb{C}[[h]]} N)/h^n(M \otimes_{\mathbb{C}[[h]]} N)$.

The most important class of $\mathbb{C}[[h]]$-modules for us are *topologically free* $\mathbb{C}[[h]]$-modules. Starting with a complex vector space V we construct $V[[h]]$ as the $\mathbb{C}[[h]]$-module consisting of formal power series $\sum_{i=0\ldots\infty} v_i h^i$, with the $\mathbb{C}[[h]]$-action given by the natural extension of the algebra structure of $\mathbb{C}[[h]]$. Just as for $\mathbb{C}[[h]]$, $V[[h]]$ is separated and complete in the h-adic topology. In fact $V[[h]]$ is precisely the h-adic completion of $V \otimes \mathbb{C}[[h]]$.

DEFINITION 2.2.7. A $\mathbb{C}[[h]]$-module will be called topologically free if it is of the form $V[[h]]$ for some complex vector space V.

A $\mathbb{C}[[h]]$-module M is said to be *torsion free* if for any $m \in M$ $hm = 0 \Rightarrow m = 0$. Notice that any topologically free module is torsion free and that if M and N are isomorphic $\mathbb{C}[[h]]$ modules then M is torsion free if and only if N is torsion free. Combined with the following result, this gives an important means of identifying topologically free modules.

THEOREM 2.2.8. *A $\mathbb{C}[[h]]$-module M is topologically free if and only if M is separated, complete and torsion free.*

In particular, suppose we have two topologically free modules, $V[[h]]$ and $W[[h]]$, then it is standard that their direct sum $V[[h]] \oplus W[[h]]$ is separated and complete, furthermore we can easily demonstrate that $V[[h]] \oplus W[[h]] \cong (V \oplus W)[[h]]$ as $\mathbb{C}[[h]]$-modules so that as $(V[[h]] \oplus W[[h]]) \bmod h$ can be identified with $V \oplus W$ we can write $V[[h]] \oplus W[[h]] = (V \oplus W)[[h]]$. A similar statement applies for the completed tensor product $V[[h]]\hat{\otimes}W[[h]]$. It is by definition a completion and so separated and complete. It is then a standard procedure to demonstrate that $V[[h]]\hat{\otimes}W[[h]] \cong (V \otimes W)[[h]]$ as $\mathbb{C}[[h]]$-modules so that we can identify $V[[h]]\hat{\otimes}W[[h]] = (V \otimes W)[[h]]$.

These and further desirable properties of topologically free modules are collected in the following theorem.

THEOREM 2.2.9. *Suppose U, V and W are complex vector spaces. Then*

(i)

$$\mathrm{Hom}_{\mathbb{C}[[h]]}(U[[h]], V[[h]]) \cong \mathrm{Hom}(U, V)[[h]]. \tag{2.2.6}$$

so that for any $\tilde{f} \in \mathrm{Hom}_{\mathbb{C}[[h]]}(U[[h]], V[[h]])$ *we identify* $\tilde{f} \mapsto \sum_{n\geq 0} f_n h^n$ *where* $f_n \in \mathrm{Hom}(U, V)$ *for all* $n \geq 0$ *and*

$$\tilde{f}(\sum_{n\geq 0} u_n h^n) = \sum_{n\geq 0}(\sum_{p+q=n} f_p(u_q))h^n \tag{2.2.7}$$

where $\sum_{n\geq 0} u_n h^n \in U[[h]]$. *We write* $\tilde{f} = (f_n)$.

(ii)

$$U[[h]] \oplus V[[h]] = (U \oplus V)[[h]]. \tag{2.2.8}$$

(iii)

$$U[[h]]\hat{\otimes} V[[h]] = (U \otimes V)[[h]]. \tag{2.2.9}$$

(iv)

$$\mathrm{Hom}^{(2)}_{\mathbb{C}[[h]]}(U[[h]], V[[h]]; W[[h]]) \cong \mathrm{Hom}_{\mathbb{C}[[h]]}((U \otimes V)[[h]], W[[h]]) \tag{2.2.10}$$

so that for any $\tilde{f} \in \mathrm{Hom}^{(2)}_{\mathbb{C}[[h]]}(U[[h]], V[[h]]; W[[h]])$ *we identify* $\tilde{f} \mapsto \sum_{n\geq 0} f_n h^n$ *where*

$$\tilde{f}(\sum_{p\geq 0} u_p h^p, \sum_{q} v_q h^q) = \sum_{n\geq 0}(\sum_{p+q+r=n} f_r(u_p, v_q))h^n, \tag{2.2.11}$$

where $f_n \in \mathrm{Hom}^{(2)}(U, V; W)$ *for all* $n \geq 0$.

If $\{v_i\}_{i\in I}$ is a basis for a complex vector space V, then the $\mathbb{C}[[h]]$-module generated by these basis elements is dense in $V[[h]]$ and we call $\{v_i\}_{i\in I}$ a topological basis for $V[[h]]$. When V is finite dimensional, $V[[h]]$ is said to be of *finite rank* and indeed is precisely the $\mathbb{C}[[h]]$-module generated by the finite basis $\{v_i\}$.

Sometimes we are interested in the quotient of a topologically free module $V[[h]]$ by the closure $\hat{I}$ of some sub-module I. $V[[h]]/\hat{I}$ inherits the metric space topology of $V[[h]]$ and is complete since we quotient by the *closure* of I. For $V[[h]]/\hat{I}$ to be topologically free it is then necessary and sufficient that it be torsion free, that is for any $v_h \in V[[h]]$ $hv_h \in \hat{I} \Rightarrow v_h \in \hat{I}$.

2.3. Topological Hopf algebras and quantised universal enveloping algebras

The topological setting of the previous section requires a modified definition of the notion of a Hopf algebra.

DEFINITION 2.3.1. A *topological Hopf algebra*, A_h, is a sextuple $A_h = (A_h, \tilde{m}, \tilde{\eta}, \tilde{\Delta}, \tilde{\epsilon}, \tilde{S})$ such that A_h is a $\mathbb{C}[[h]]$-module with $\mathbb{C}[[h]]$-linear maps, $\tilde{m} : A_h \hat{\otimes} A_h \to A_h$, $\tilde{\eta} : \mathbb{C}[[h]] \to A_h$, $\tilde{\Delta} : A_h \to A_h \hat{\otimes} A_h$, $\tilde{\epsilon} : A_h \to \mathbb{C}[[h]]$ and $\tilde{S} : A_h \to A_h$ satisfying the same relations as (1.6.5) but with the tensor products replaced by completed tensor products.

Very many examples of what are called quantum groups are examples of the following structure, invented by Drinfeld.

DEFINITION 2.3.2. A *topological quasitriangular Hopf algebra* $(A_h, \mathcal{R})$ is a topological Hopf algebra A_h together with an invertible element $\mathcal{R}$ of the completed tensor product, $A_h \hat{\otimes} A_h$, such that

$$\Delta^{\mathrm{op}}(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}, \tag{2.3.1}$$

$$(\Delta \hat{\otimes} \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \tag{2.3.2}$$

$$(\mathrm{id} \hat{\otimes} \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}, \tag{2.3.3}$$

where $\Delta^{\mathrm{op}}(a) = a_{(2)} \hat{\otimes} a_{(1)}$ and if $\mathcal{R} = \sum_i R_i \hat{\otimes} R'_i$ then for example $\mathcal{R}_{13} = \sum_i R_i \hat{\otimes} 1 \hat{\otimes} R'_i$. In this case

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{2.3.4}$$

which is the celebrated *quantum Yang-Baxter equation* (QYBE).

If our topological quasitriangular Hopf algebra $(A_h, \mathcal{R})$ is such that A_h is topologically free as a $\mathbb{C}[[h]]$-module, then the underlying vector space is $(A_h/hA_h)[[h]]$ and we can legitimately think about the topological Hopf and quasitriangular structures in terms of formal power series. Thus the Hopf maps $\tilde{m}$, $\tilde{\eta}$, $\tilde{\Delta}$, $\tilde{\epsilon}$ and $\tilde{S}$ may be treated (in view of Theorem 2.2.9) in terms of the corresponding families of $\mathbb{C}$-linear maps (m_n), (η_n), (Δ_n), (ϵ_n) and (S_n). In particular, the coproduct Δ is determined by its action on elements $a \in A_h/hA_h$ as $\Delta(a) = \sum_{n\geq 0} \Delta_n(a)h^n$ where the Δ_n are $\mathbb{C}$-linear maps such that $\Delta_n : A_h/hA_h \to (A_h/hA_h) \otimes (A_h/hA_h)$ and the universal R-matrix may be written as $\mathcal{R} = \sum_{n\geq 0} \mathcal{R}_n h^n$ where the $\mathcal{R}_n$ are elements of $(A_h/hA_h) \otimes (A_h/hA_h)$. This makes for a considerable conceptual simplification so it is rather fortunate that quantum groups are indeed topologically free!

DEFINITION 2.3.3. A *quantised universal enveloping algebra* (QUEA) corresponding to a Lie algebra $\mathfrak{g}$ is a topological Hopf algebra $U_h(\mathfrak{g})$ such that $(U_h(\mathfrak{g})/hU_h(\mathfrak{g})) \cong U(\mathfrak{g})$ as Hopf algebras, we can identify $U_h(\mathfrak{g})$ with $U(\mathfrak{g})[[h]]$ as $\mathbb{C}[[h]]$-modules and $\eta_n = 0$ and $\epsilon_n = 0$ for all $n > 0$. Thus we denote a QUEA by $U_h(\mathfrak{g}) = (U(\mathfrak{g})[[h]], \tilde{m}, \eta, \tilde{\Delta}, \epsilon, \tilde{S})$. A *quasitriangular QUEA* is a QUEA which is also a topological quasitriangular Hopf algebra such that $\mathcal{R} = 1 \otimes 1 \bmod h$.

REMARK 2.3.4. Notice that the unit and counit maps of a QUEA corresponding to $\mathfrak{g}$ are just the $\mathbb{C}[[h]]$-linear extensions of the corresponding maps of $U(\mathfrak{g})$. In this case we will denote them simply as η and ϵ respectively. We will see an 'explanation' of this fact in the following section.

Two basic constructions will be the following:

DEFINITION 2.3.5. Denote by $\mathbb{C}\{X\}$ the usual $\mathbb{C}$-algebra freely generated by the symbols $X = \{X_1, X_2, \dots X_n\}$. Then the *topologically free algebra generated by the symbols* X is $(\mathbb{C}\langle X\rangle)[[h]]$.

DEFINITION 2.3.6. An algebra A_h is *topologically generated by the symbols X subject to the relations I_i* if $A_h = (\mathbb{C}\langle X\rangle)[[h]]/\hat{I}$ where $\hat{I}$ is the closure in the h-adic topology of the ideal in $(\mathbb{C}\langle X\rangle)[[h]]$ corresponding to the relations I_i.

REMARK 2.3.7. By factoring out the *closure* of I we ensure that infinite sums, $\sum_{i=0\dots\infty} a_i h^i$ with all $a_i \in I$ are not left in the quotient. If they were then the zero Cauchy sequence $(s_n) + I$ where $s_n = \sum_{i=0\dots n} a_i h^i$ would converge to 0 *and* $\sum_{i=0\dots\infty} a_i h^i + I$ so that the resulting algebra would *not* be complete.

2.4. Deformations of Hopf algebras

Quantised universal enveloping algebras are formal deformations of the classical universal enveloping algebras according to the following definition.

DEFINITION 2.4.1. A *formal deformation* of a Hopf algebra $A = (A, m, \eta, \Delta, \epsilon, S)$ over $\mathbb{C}$, is a topological Hopf algebra over $\mathbb{C}[[h]]$, $A_h = (A[[h]], \tilde{m}, \tilde{\eta}, \tilde{\Delta}, \tilde{\epsilon}, \tilde{S})$. As indicated, A_h is the topologically free $\mathbb{C}[[h]]$-module $A[[h]]$ but we must also have $m = \tilde{m} \bmod h$ and $\Delta = \tilde{\Delta} \bmod h$.

Recalling the identification $\tilde{f} = (f_n)$, a formal deformation A_h is said to have a *trivial algebra structure* if $m_n = 0$ for all $n > 0$ $(m_0 = m)$ and a *trival coalgebra structure* if $\Delta_n = 0$ for all $n > 0$ $(\Delta_0 = \Delta)$. If a formal deformation has both a trivial algebra structure and a trivial coalgebra structure then it is called simply a trivial deformation.

If A_h and A'_h are two formal deformations of a Hopf algebra A then they are said to be equivalent if there exists an isomorphism of Hopf algebras $\tilde{\phi} : A_h \to A'_h$ such that $\tilde{\phi} = \mathrm{id} \bmod h$.

THEOREM 2.4.2. *Any formal deformation of a Hopf algebra A is equivalent to one whose unit $\tilde{\eta}$ and counit $\tilde{\epsilon}$ are such that $\eta_n = 0$ and $\epsilon_n = 0$ for all $n > 0$. Any formal deformation of a Hopf algebra treated as a bialgebra is a topological Hopf algebra.*

Let us recall that if A is an algebra over $\mathbb{C}$ and M is an A-bimodule with left and right actions $\triangleright$ and $\triangleleft$ respectively, then by an *n-cochain*, we mean an n-linear map $f : A^{\otimes n} \to M$. We define the *coboundary* of an n-cochain f to be the $(n+1)$-cochain $\mathrm{d}^n f$ given by

$$\mathrm{d}^n f(a_1, a_{2,}, \dots, a_{n+1}) = a_1 \triangleright f(a_2, \dots, a_{n+1}) \tag{2.4.1}$$

$$+ \sum_{i=1}^{n} (-1)^i f(a_1, \dots, a_i a_{i+1}, \dots, a_{n+1}) \tag{2.4.2}$$

$$+ (-1)^{n+1} f(a_1, \dots, a_n) \triangleleft a_{n+1}, \tag{2.4.3}$$

so that $\mathrm{d}^{n+1} \circ \mathrm{d}^n = 0$. An *$n$-cocycle* is then an n-cochain such that $\mathrm{d}^n f = 0$ whilst an *n-coboundary* is an n-cochain of the form $\mathrm{d}^{n-1} g$ for some $(n-1)$-cochain g. Denoting the set of all n-cocycles by $Z^n(A, M)$ and the set of all n-coboundaries by $B^n(A, M)$, the quotient space

$$H^n(A, M) = Z^n(A, M)/B^n(A, M) \tag{2.4.4}$$

is called the *n-th Hochschild cohomology group*.

Given an arbitrary Hopf algebra A, the deformed multiplication, $\tilde{m}$, and coproduct, $\tilde{\Delta}$, are restricted by the requirements that in any extant deformation of A they must be respectively associative and co-associative and also compatible in the sense that $\tilde{\Delta}$ is an algebra map. For the multiplication this means that for all $a, b, c \in A$

$$\tilde{m}(\tilde{m}(a, b), c) = \tilde{m}(a, \tilde{m}(b. c)), \tag{2.4.5}$$

so that

$$\sum_{n \geq 0} (\sum_{p+q=n} m_p(m_q(a, b), c)) h^n = \sum_{n \geq 0} (\sum_{p+q=n} m_p(a. m_q(b, c))) h^n. \tag{2.4.6}$$

and we obtain an infinite number of constraints on the $\mathbb{C}$-linear maps (m_n)

$$\sum_{p+q=n} m_p(m_q(a,b),c) = \sum_{p+q=n} m_p(a, m_q(b,c)), \tag{2.4.7}$$

for all $n > 0$. Similarly we derive constraints on the Δ_n from coassociativity and constraints on the interrelations between the m_n and Δ_n from the compatibility condition.

In particular, for the multiplication, at order h $(n = 1)$ we have

$$m_1(a,b)c + m_1(ab,c) = m_1(a,bc) + am_1(b,c), \tag{2.4.8}$$

and this is precisely the condition for m_1 to be a 2-cocycle in $Z^2(A, A)$ for the Hochschild cohomology theory of the *algebra* A. From the co-associativity condition we find that Δ_1 must be a 2-cocycle in a 'dualised' Hochschild cohomology theory for the *coalgebra* A. In combination with the compatibility condition, we have the condition that (m_1, Δ_1) be a 2-cocycle in the *Gerstenhaber-Schack cohomology* theory of A as a *bialgebra*.

We must identify equivalent deformations A_h and A'_h. If the isomorphism is given by $\tilde{\phi}$ then we obtain, for the multiplication, the system of constraints

$$\sum_{p+q+r=n} m'_p(\phi_q(a), \phi_r(b)) = \sum_{p+q=n} \phi_p(m_q(a,b)) \tag{2.4.9}$$

for all $a, b \in A$ and $n > 0$. To order h $(n = 1)$ this gives us

$$m_1(a,b) - m'_1(a,b) = \phi_1(a)b + a\phi_1(b) - \phi_1(ab), \tag{2.4.10}$$

which tells us that for equivalent deformations, m_1 and m'_1 differ by a 2-coboundary in $B^2(A, A)$ in the Hochschild cohomology theory of A as an algebra. Again we obtain similar results for the coproduct, relevant in the context of the Gerstenhaber-Schack cohomology.

Thus the space of inequivalent deformations of A, to order h, treated as an algebra, is precisely the second *Hochschild cohomology group* $H^2(A, A) = Z^2(A, A)/B^2(A, A)$. Similarly we find that the space of inequivalent deformations of A to order h as a coalgebra is the second dualised Hochschild cohomology group $H^2_{coalg}(A, A)$ and the space of bialgebra deformations of H to order h is the second order *Gerstenhaber-Schack cohomology group* $H^2_{GS}(A, A)$.

THEOREM 2.4.3. *If $H^2(A, A) = 0$ then every formal deformation of A is equivalent to one which has a trivial algebra structure. If $H^2_{coalg}(A, A) = 0$ then every formal deformation of A is equivalent to one that has a trivial coalgebra structure. If $H^2_{GS}(A, A) = 0$ then every formal deformation of A is equivalent to a trivial one.*

The following result is one of two key cohomological results for the subject of quantum groups. It tells us that for the universal enveloping algebras, $U(\mathfrak{g})$, of complex semi-simple Lie algebras, $\mathfrak{g}$, there are *no* non-trivial deformations of $U(\mathfrak{g})$ as an *algebra* but there *may* be Hopf algebra deformations.

THEOREM 2.4.4. *For any complex simple Lie algebra* $\mathfrak{g}$

$$H^2(U(\mathfrak{g}), U(\mathfrak{g})) = 0, \quad H^2_{GS}(U(\mathfrak{g}), U(\mathfrak{g})) = \wedge^2(\mathfrak{g}). \tag{2.4.11}$$

This result leads directly to the following theorem.

THEOREM 2.4.5. *Let $\mathfrak{g}$ be a complex simple Lie algebra and corresponding to it a QUEA $U_h(\mathfrak{g}) = (U(\mathfrak{g})[[h]], \tilde{m}, \eta, \tilde{\Delta}, \epsilon, \tilde{S})$. Then there is an isomorphism of Hopf algebras, $\tilde{\phi} : U_h(\mathfrak{g}) \to U_h(\mathfrak{g})_\phi$ such that $\tilde{\phi} = \mathrm{id} \bmod h$, where*

$$U_h(\mathfrak{g})_\phi = (U(\mathfrak{g})[[h]], m, \eta, \tilde{\Delta}_\phi, \eta, \tilde{S}_\phi) \tag{2.4.12}$$

is a topological Hopf algebra on $U(\mathfrak{g})[[h]]$ with trivial multiplication and

$$\tilde{\Delta}_\phi = (\tilde{\phi} \hat{\otimes} \tilde{\phi}) \circ \tilde{\Delta} \circ \tilde{\phi}^{-1} \tag{2.4.13}$$

and

$$\tilde{S}_\phi = \tilde{\phi} \circ \tilde{S} \circ \tilde{\phi}^{-1}. \tag{2.4.14}$$

When $U_h(\mathfrak{g})$ is quasitriangular with universal R-matrix $\mathcal{R}$, so is $U_h(\mathfrak{g})_\phi$ with universal R-matrix $\mathcal{R}^\phi = (\tilde{\phi} \hat{\otimes} \tilde{\phi})(\mathcal{R})$.

REMARK 2.4.6. Theorem 2.4.4 tells us that $U_h(\mathfrak{g})$ must be isomorphic as an *algebra* to a deformation of $U(\mathfrak{g})$ which is trivial as an algebra. This isomorphism, $\tilde{\phi}$, is then used to define the coalgebra structures on $U(\mathfrak{g})[[h]]$ such that the algebra isomorphism extends to a Hopf algebra isomorphism. Note that it is not obvious that $\epsilon \circ \tilde{\phi}^{-1} = \epsilon$. However this may be proved using a relatively simple cohomological argument.

Dually, a similar result holds for the cohomology groups associated with deformations of the coordinate rings of simple complex linear algebraic groups. In this case however the roles of algebra and coalgebra are reversed and there may again be Hopf algebra deformations but there can be *no* non-trivial deformations of $\mathbb{C}[G]$ as a *coalgebra.*

We should note that these cohomology theory results inform us of the *possibility* of deformations. They do not tell us how many (if any) of the first order deformations extend to full formal deformations. We say that a structure is *rigid* with respect to deformations if the cohomology theory tells us that no formal deformations are possible.

2.5. The Drinfeld-Jimbo quantum groups

For any complex simple Lie algebra $\mathfrak{g}$ of rank l we define integers $d_1, d_2, \ldots, d_l$ as follows. In the cases A_l, D_l, E_6, E_7, E_8 $d_i = 1$ for all i. When $\mathfrak{g} = B_l$, $d_i = 1$ for $1 \le i \le l-1$ and $d_l = 2$. When $\mathfrak{g} = C_l$, set $d_i = 2$ for $1 \le i \le l-1$ and $d_l = 1$. For F_4, $d_1 = d_2 = 1$ and $d_3 = d_4 = 2$ while for G_2, $d_1 = 3$ and $d_2 = 1$. Also, for any positive integers $m \ge n$, define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = [n]_q[n-1]_q \ldots [1]_q, \tag{2.5.1}$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!}. \tag{2.5.2}$$

We can now state the fundamental definition.

DEFINITION 2.5.1. The *Drinfeld-Jimbo* algebra $U_h(\mathfrak{g})$ is the algebra over $\mathbb{C}[[h]]$ topologically generated by the $3l$ symbols $\{X_i, Y_i, H_i\}_{i=1}^{l}$ subject to the relations

$$[H_i, H_j] = 0, \tag{2.5.3}$$

$$[H_i, X_j] = A_{ji}X_j, \quad [H_i, Y_j] = -A_{ji}Y_j, \tag{2.5.4}$$

$$[X_i, Y_i] = \delta_{ij}\frac{\sinh d_i h H_i/2}{\sinh d_i h/2}, \tag{2.5.5}$$

for all $i, j = 1 \dots l$ together with

$$\sum_{k=0}^{1-A_{ji}} (-1)^k \begin{bmatrix} 1-A_{ji} \\ k \end{bmatrix}_{e^{d_i h/2}} (Z_i)^s Z_j (Z_i)^t = 0, \tag{2.5.6}$$

for $i \neq j$ and $Z_i = X_i$ and Y_i.

In neat correspondence with the classical case there is a PBW basis for $U_h(\mathfrak{g})$ which is constructed using the Weyl group of $\mathfrak{g}$.

THEOREM 2.5.2. *$U_h(\mathfrak{g})$ is a topological Hopf algebra with Hopf maps defined on the generators for all $i = 1 \dots l$ by*

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i, \tag{2.5.7}$$

$$\Delta(X_i) = X_i \otimes e^{d_i h H_i/4} + e^{-d_i h H_i/4} \otimes X_i, \quad \Delta(Y_i) = Y_i \otimes e^{d_i h H_i/4} + e^{-d_i h H_i/4} \otimes Y_i, \tag{2.5.8}$$

$$\epsilon(H_i) = \epsilon(X_i) = \epsilon(Y_i) = 0, \tag{2.5.9}$$

$$S(H_i) = -H_i, \quad S(X_i) = -e^{d_i h H_i/2} X_i, \quad S(Y_i) = -e^{-d_i h H_i/2} Y_i, \tag{2.5.10}$$

extended in the cases Δ and ϵ as $\mathbb{C}[[h]]$-algebra maps and for S as a $\mathbb{C}[[h]]$-antialgebra map. $U_h(\mathfrak{g})$ is isomorphic as a $\mathbb{C}[[h]]$-module to $U(\mathfrak{g})[[h]]$ and indeed is a QUEA.

The fact that $U_h(\mathfrak{g})$ may be regarded as being built on the vector space $U(\mathfrak{g})[[h]]$ provides some justification for using the same symbols as classically. However we must be careful since the quantum and classical symbols certainly cannot be identified as Hopf algebra elements. On the other hand it *is* true mod$h$ that we can identify them as Hopf algebraic elements and we have already seen that there exists an isomorphism $\tilde{\phi} : U_h(\mathfrak{g}) \to U_h(\mathfrak{g})_\phi$. In fact Drinfeld proved more for the Drinfeld-Jimbo quantum groups.

THEOREM 2.5.3. *Let $\mathfrak{g}$ be a complex simple Lie algebra and let $U_h(\mathfrak{g})$ be the Drinfeld-Jimbo QUEA corresponding to $\mathfrak{g}$. Denote by $\mathfrak{h}_h$ the complex span of $\{H_i\}_{i=1}^{l}$ in $U_h(\mathfrak{g})$. Then we may choose the isomorphism $\tilde{\phi} : U_h(\mathfrak{g}) \to U_h(\mathfrak{g})_\phi$ such that $\tilde{\phi}|_{\mathfrak{h}_h} = \mathrm{id}$.*

This, together with Theorem 2.4.5 indicates that the representation theory of $U_h(\mathfrak{g})$ will be very similar to that of $U(\mathfrak{g})$ and it implies that we really can identify the classical Cartan subalgebra elements of $\mathfrak{g}$ with the corresponding elements of $U_h(\mathfrak{g})$.

EXAMPLE 2.5.4. $U_h(\mathfrak{sl}_2(\mathbb{C}))$ is the quantised enveloping algebra topologically generated by $\{X, Y, H\}$ and the relations $[H_h, X_h] = 2X_h$, $[H_h, Y_h] = -2Y_h$ and $[X_h, Y_h] =$

$\frac{\sinh hH_h/2}{\sinh h/2}$ with the topological Hopf algebra structures defined on the generators as

$$\Delta(H) = H\hat{\otimes}1 + 1\hat{\otimes}H, \tag{2.5.11}$$

$$\Delta(X) = X\hat{\otimes}e^{hH/4} + e^{-hH/4}\hat{\otimes}X, \tag{2.5.12}$$

$$\Delta(Y) = Y\hat{\otimes}e^{hH/4} + e^{-hH/4}\hat{\otimes}Y, \tag{2.5.13}$$

$$\epsilon(H) = \epsilon(X) = \epsilon(Y) = 0, \tag{2.5.14}$$

$$S(H) = -H, \quad S(X) = -e^{h/2}X, \quad S(Y) = -e^{-h/2}Y. \tag{2.5.15}$$

These are then extended respectively as $\mathbb{C}[[h]]$-algebra, -algebra and -anti-algebra maps to the whole of $U_h(\mathfrak{sl}_2(\mathbb{C}))$. There is a PBW-type topological basis of monomials $\{Y_h^\alpha H_h^\beta X_h^\gamma : \alpha, \beta, \gamma \in \mathbb{N}\}$ for $U_h(\mathfrak{sl}_2(\mathbb{C}))$. $U_h(\mathfrak{sl}_2(\mathbb{C}))$ is moreover topologically quasitriangular with universal R-matrix

$$\mathcal{R} = \sum_{n=0}^{\infty} \frac{(e^{\frac{h}{2}} - e^{-\frac{h}{2}})^n}{[n]_{e^{h/2}}!} e^{-\frac{hn(n+1)}{4}} e^{\frac{h}{4}(H\hat{\otimes}H + n(H\hat{\otimes}1 - 1\hat{\otimes}H))}(X^n\hat{\otimes}Y^n). \tag{2.5.16}$$

The property of quasitriangularity exhibited by $U_h(\mathfrak{sl}_2(\mathbb{C}))$ is shared by *all* the Drinfeld-Jimbo quantum groups and in each case the universal R-matrix is known. Let us also note that the map $\tilde{\phi}$ is known for the example of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ but not for any other Drinfeld-Jimbo quantum group. In Section 2.10 we will see that such a map is also known for the Jordanian deformation of $U(\mathfrak{sl}_2(\mathbb{C}))$.

2.6. Representation theory of formal deformations and QUEAs

Suppose $A = (A, m, \eta, \Delta, \epsilon, S)$ is a Hopf algebra over $\mathbb{C}$ and $A_h = (A[[h]], \tilde{m}, \eta, \tilde{\Delta}, \epsilon, \tilde{S})$ is a formal deformation of A.

DEFINITION 2.6.1. A *topologically free representation* of a Hopf algebra A_h is a pair $(\tilde{\rho}^{V[[h]]}, V[[h]])$ where V is a complex vector space and $\tilde{\rho}^{V[[h]]}$ is a $\mathbb{C}[[h]]$-algebra map, $\tilde{\rho}^{V[[h]]} : A_h \to \mathrm{End}_{\mathbb{C}[[h]]}(V[[h]])$. Thus for all $a, b \in A$ and $n \geq 0$ we have $\tilde{\rho}^{V[[h]]} = (\rho_n^V)$ where

$$\sum_{p+q=n} (\rho_p^V(m_q(a,b)) - \rho_p^V(a) \circ \rho_q^V(b)) = 0, \tag{2.6.1}$$

so that in particular (ρ_0^V, V) is a representation of A. We will only consider representations of *finite rank.*

REMARK 2.6.2. In other words $V[[h]]$ is an example of the more general notion of a topological A_h-module with a $\mathbb{C}[[h]]$-linear action $\triangleright_h$ of A_h on $V[[h]]$ such that $a \triangleright_h \tilde{v} = \tilde{\rho}^{V[[h]]}(a)\tilde{v}$ for all $a \in A$ and $\tilde{v} \in V[[h]]$. We deal almost entirely with topologically free modules and representations and will refer to topologically free representations simply as *representations.*

REMARK 2.6.3. In the definition, we identify $\mathrm{End}_{\mathbb{C}[[h]]}(V[[h]]) = \mathrm{End}(V)[[h]]$ endowed with the trivial algebra structure as a formal deformation of $\mathrm{End}(V)$.

DEFINITION 2.6.4. If $(\tilde{\rho}^{V[[h]]}, V[[h]])$ and $(\tilde{\rho}^{W[[h]]}, W[[h]])$ are two representations of A_h, then a $\mathbb{C}[[h]]$-linear map $\tilde{\psi} : V[[h]] \to W[[h]]$ such that for all $a \in A$

$$\tilde{\rho}^{W[[h]]}(a) \circ \tilde{\psi} = \tilde{\psi} \circ \tilde{\rho}^{V[[h]]}(a), \tag{2.6.2}$$

is said to *intertwine* the representations. In particular if $\tilde{\psi}$ is an isomorphism then the representations are *equivalent*.

REMARK 2.6.5. A $\mathbb{C}[[h]]$-linear map $\tilde{\psi}$ between topological spaces $V[[h]]$ and $W[[h]]$ is easily seen to be surjective (respectively injective, bijective) if and only if ψ_0 is surjective (respectively injective, bijective).

We have constructions for topological representations entirely analogous to those for ordinary representations. We collect them in the following theorem.

THEOREM 2.6.6. *Suppose* $(\tilde{\rho}^{V[[h]]}, V[[h]])$ *and* $(\tilde{\rho}^{W[[h]]}, W[[h]])$ *are representations of* A_h, *then* $V[[h]] \oplus W[[h]] = (V \oplus W)[[h]]$ *carries a representation of* A_h, $(\tilde{\rho}^{(V\oplus W)[[h]]}, (V \oplus W)[[h]])$, *called the* ***direct sum representation***, *such that for all* $a \in A$ *and* $\tilde{v} \in V[[h]]$, $\tilde{w} \in W[[h]]$

$$\tilde{\rho}^{(V\oplus W)[[h]]}(a)(\tilde{v} + \tilde{w}) = \tilde{\rho}^{V[[h]]}(a)\tilde{v} + \tilde{\rho}^{W[[h]]}(a)\tilde{w}. \tag{2.6.3}$$

The completed tensor product, $V[[h]]\hat{\otimes}W[[h]] = (V \otimes W)[[h]]$ *carries a representation called the* ***tensor product representation***, $(\tilde{\rho}^{(V\otimes W)[[h]]}, (V \otimes W)[[h]])$, *such that for all* $a \in A$

$$\tilde{\rho}^{(V\otimes W)[[h]]}(a) = (\tilde{\rho}^{V[[h]]}\hat{\otimes}\tilde{\rho}^{W[[h]]})(\tilde{\Delta}(a)), \tag{2.6.4}$$

that is, for all $n \geq 0$

$$\rho_n^{V\otimes W}(a) = \sum_{p+q+r=n} (\rho_p^V \otimes \rho_q^W)(\Delta_r(a)). \tag{2.6.5}$$

The $\mathbb{C}[[h]]$*-linear dual of* $V[[h]]$ *is* $V^*[[h]]$ *and carries a representation,* $(\tilde{\rho}^{V^*[[h]]}, V^*[[h]])$, *the* ***dual*** *or* ***contragredient representation***, *such that for all* $a \in A$

$$\tilde{\rho}^{V^*[[h]]}(a) = (\tilde{\rho}^{V[[h]]}(\tilde{S}(a)))^t, \tag{2.6.6}$$

or, for all $n \geq 0$

$$\rho_n^{V^*}(a) = \sum_{p+q=n} (\rho_p^V(S_q(a)))^t. \tag{2.6.7}$$

REMARK 2.6.7. Notice that in each case the definition reduces to the ordinary one mod$h$.

Suppose now that we have a formal deformation A_h of A which has a trivial algebra structure. Then if $(\tilde{\rho}^{V[[h]]}, V[[h]])$ is a representation of A_h we have

$$\rho_n^V(ab) = \sum_{p+q=n} \rho_p^V(a) \circ \rho_q^V(b), \tag{2.6.8}$$

for all $a, b \in A$ and $n \geq 0$. In this circumstance, the representation $(\tilde{\rho}^{V[[h]]}, V[[h]])$ is said to be a *formal deformation of the representation* (ρ_0, V) *of* A. In particular, if $\rho_n^V = 0$ for all $n > 0$ then $(\tilde{\rho}^{V[[h]]}, V[[h]])$ is said to be a *trivial deformation of* (ρ_0^V, V) and written $(\rho_0^V, V[[h]])$.

THEOREM 2.6.8. *Suppose* A_h *is a formal deformation, with a trivial algebra structure, of a Hopf algebra* A *and that* $(\tilde{\rho}^{V[[h]]}, V[[h]])$ *is a representation of* A_h. *Then* $H^1(A, \mathrm{End}\, V) = 0$ *implies that* $(\tilde{\rho}^{V[[h]]}, V[[h]])$ *is equivalent, by an intertwiner* $\tilde{\psi} = \mathrm{id}\, \mathrm{mod} h$, *to the trivial representation* $(\rho_0^V, V[[h]])$.

The second key cohomological result is presented in the following theorem. It tells us that for complex simple Lie algebras, $\mathfrak{g}$, *every* formal deformation of a representation (ρ^V, V) of $U(\mathfrak{g})$ must be equivalent to the trivial deformation $(\rho^V, V[[h]])$ obtained by extending (ρ^V, V) $\mathbb{C}[[h]]$-linearly.

THEOREM 2.6.9. *For any complex simple Lie algebra $\mathfrak{g}$ and finite dimensional representation (ρ^V, V) of $U(\mathfrak{g})$*

$$H^1(U(\mathfrak{g}), \operatorname{End} V) = 0. \tag{2.6.9}$$

REMARK 2.6.10. This result follows as a particular consequence of the Whitehead lemmas: *If $\mathfrak{g}$ is a complex simple Lie algebra and V is any finite dimensional left $\mathfrak{g}$-module, then $H^1(\mathfrak{g}, V) = H^2(\mathfrak{g}, V) = 0$.* In these cases the cohomology is the Chevalley cohomology.

Let us denote by $\operatorname{Rep}(U_h(\mathfrak{g}))$, $\operatorname{Rep}(U_h(\mathfrak{g})_\phi)$ and $\operatorname{Rep}(U(\mathfrak{g}))$ the sets of equivalence classes of representations of $U_h(\mathfrak{g})$, $U_h(\mathfrak{g})_\phi$ and $U(\mathfrak{g})$ respectively.

Consider the relationship between $\operatorname{Rep}(U_h(\mathfrak{g})_\phi)$ and $\operatorname{Rep}(U(\mathfrak{g}))$. If (ρ^V, V) is a representation of $U(\mathfrak{g})$ then we can extend this $\mathbb{C}[[h]]$-linearly to a representation $(\rho^V, V[[h]])$ of $U_h(\mathfrak{g})_\phi$. Clearly if (π^W, W) is equivalent as a representation of $U(\mathfrak{g})$ to (ρ^V, V) then $(\pi^W, W[[h]])$ is equivalent as a representation of $U_h(\mathfrak{g})_\phi$ to $(\rho^V, V[[h]])$. Conversely, given any representation $(\tilde{\rho}^{V[[h]]}, V[[h]])$ of $U_h(\mathfrak{g})_\phi$ we obtain a representation (ρ_0^V, V) of $U(\mathfrak{g})$ where $\tilde{\rho}^{V[[h]]} = \sum_{n\geq 0} \rho_n^V h^n$. That the correspondence we have set up between $\operatorname{Rep}(U_h(\mathfrak{g})_\phi)$ and $\operatorname{Rep}(U(\mathfrak{g}))$ is bijective now follows since any representation of $U_h(\mathfrak{g})_\phi$ which is equivalent mod$h$ to the representation (ρ_0^V, V) must, by Theorem 2.6.9, be equivalent to $(\rho_0^V, V[[h]])$. It should be clear that this bijection preserves direct sums, tensor products and duals. In particular, suppose $(\tilde{\rho}^{(V\otimes W)[[h]]}, (V \otimes W)[[h]])$ is a tensor product representation of $U_h(\mathfrak{g})_\phi$, then for all $x \in U(\mathfrak{g})$

$$\tilde{\rho}^{(V\otimes W)[[h]]}(x) = (\tilde{\rho}^{V[[h]]} \hat{\otimes} \tilde{\rho}^{W[[h]]})(\tilde{\Delta}_\phi(x)), \tag{2.6.10}$$

and the corresponding representation of $U(\mathfrak{g})$ is $(\rho_0^{V\otimes W}, V \otimes W)$. Moreover $(\tilde{\rho}^{(V\otimes W)[[h]]}, (V \otimes W)[[h]])$ *must be equivalent* to the representation $(\rho_0^{V\otimes W}, (V \otimes W)[[h]])$ of $U_h(\mathfrak{g})_\phi$ where for all $x \in U(\mathfrak{g})$

$$\rho_0^{V\otimes W}(x) = (\rho_0^V \otimes \rho_0^W)(\Delta(x)), \tag{2.6.11}$$

and Δ is the *usual* coproduct of $U(\mathfrak{g})$ $\mathbb{C}[[h]]$-linearly extended to the space $U(\mathfrak{g})[[h]]$ equipped with the trivial algebra structure.

If $(\tilde{\rho}^{V[[h]]}, V[[h]])$ is a representation of $U_h(\mathfrak{g})$ then $(\tilde{\rho}^{V[[h]]}(\tilde{\phi}^{-1}(\cdot)), V[[h]])$ is a representation of $U_h(\mathfrak{g})_\phi$. Conversely, given any representation of $U_h(\mathfrak{g})_\phi$ we have a representation of $U_h(\mathfrak{g})$ by first using the isomorphism $\tilde{\phi}$. It is obvious that this correspondence between representations of $U_h(\mathfrak{g})$ and representations of $U_h(\mathfrak{g})_\phi$ preserves equivalence, direct sums, tensor products and duals.

Thus we have obtained the following result:

THEOREM 2.6.11. *There is a bijective correspondence between* $\operatorname{Rep}(U_h(\mathfrak{g}))$ *and* $\operatorname{Rep}(U(\mathfrak{g}))$ *which preserves direct sums, tensor products and duals. In this correspondence the equivalence class of a representation $(\tilde{\rho}^{V[[h]]}, V[[h]])$ of $U_h(\mathfrak{g})$ is mapped to the equivalence class of* *the representation (ρ_0^V, V) of $U(\mathfrak{g})$ while conversely the equivalence class of a representation (ρ^V, V) of $U(\mathfrak{g})$ is mapped to the equivalence class of the representation $(\rho^V \circ \tilde{\phi}, V[[h]])$ of $U_h(\mathfrak{g})$. In particular any representation $(\tilde{\rho}^{V[[h]]}, V[[h]])$ of $U_h(\mathfrak{g})$ is equivalent to the representation $(\rho_0^V(\tilde{\phi}(\cdot)), V[[h]])$ of $U_h(\mathfrak{g})$.*

Suppose we have a representation $(\tilde{\rho}^{V[[h]]}, V[[h]])$ of $U_h(\mathfrak{g})$. When we talk about $U_h(\mathfrak{g})$-*submodules* of a topological module such as $V[[h]]$, we will mean *closed* submodules. Then all $U_h(\mathfrak{g})$-submodules of a topologically free module are toplogically free. A rather trivial collection of $U_h(\mathfrak{g})$-submodules of $V[[h]]$ is given by $h^n V[[h]] \subset V[[h]]$ for any $n > 0$. They are trivial in the sense that they are all isomorphic as $U_h(\mathfrak{g})$-modules to $V[[h]]$ itself. They are topologically free with $h^n V[[h]] = W[[h]]$ for $W = h^n V$ where $h^n V = \{vh^n \mid v \in V\}$ is of course isomorphic to V. However since $h^n V[[h]] \subset V[[h]]$, $(\rho^V, V[[h]])$ cannot be irreducible even when V is irreducible. Rather, we will see that in this case $V[[h]]$ is *indecomposable*; that is, it cannot be identified as a direct sum of non-trivial submodules.

Suppose we know that a representation $(\tilde{\rho}^{V[[h]]}, V[[h]])$ of $U_h(\mathfrak{g})$ contains no submodules other than ones of the form $h^n V[[h]]$, then certainly $V[[h]]$ is indecomposable. If instead $V[[h]]$ does contain a submodule which is not of the form $h^n V[[h]]$ then denote it by W. Any element $\tilde{w} \in W$ may be written as $\tilde{w} = \sum_{n \geq 0} w_n h^n$ so take r to be the smallest number such that W_r, the set of w_r for all $\tilde{w} \in W$, is non-zero. Then it is not difficult to see that W_r is a $U(\mathfrak{g})$-submodule of V and that $W = (h^r W_r)[[h]]$. Then there exists another $U(\mathfrak{g})$-module, W_r' say, such that $V = W_r \oplus W_r'$. But then, by Theorem 2.6.11, $(\tilde{\rho}^{V[[h]]}, V[[h]])$ must be equivalent to the representation $(\rho^{W_r \oplus W_r'} \circ \tilde{\phi}, (W_r \oplus W_r')[[h]])$ with both $W_r[[h]]$ and $W_r'[[h]]$, $U_h(\mathfrak{g})$-submodules of $V[[h]]$. Thus we see that a $U_h(\mathfrak{g})$-module $V[[h]]$ is indecomposable if and only if the only submodules of $V[[h]]$ are of the form $h^n V[[h]]$. Moreover it is clear from this argument that $V[[h]]$ is indecomposable if and only if V is irreducible.

The following theorem is a consequence of the preceding discussion:

THEOREM 2.6.12. *The bijective correspondence between* $\mathrm{Rep}(U_h(\mathfrak{g}))$ *and* $\mathrm{Rep}(U(\mathfrak{g}))$ *preserves submodules and indecomposability of modules and every $U_h(\mathfrak{g})$-module can be written as a direct sum of indecomposable $U_h(\mathfrak{g})$-submodules.*

An obvious but very important consequence of Theorems 2.6.11 and 2.6.12 is the preservation of tensor decompositions:

THEOREM 2.6.13. *Consider irreducible $U(\mathfrak{g})$-modules $V(\Lambda')$ and $V(\Lambda'')$ and define non-negative integers $n^{\Lambda}_{\Lambda',\Lambda''}$ through their appearance in the decomposition of the tensor product $V(\Lambda) \otimes V(\Lambda')$ into irreducible submodules $V(\Lambda)$ according to*

$$V(\Lambda') \otimes V(\Lambda'') \cong \oplus_{\Lambda \in P^+} n^{\Lambda}_{\Lambda',\Lambda''} V(\Lambda). \tag{2.6.12}$$

Then the decompositions of the corresponding indecomposable $U_h(\mathfrak{g})$-modules $V(\Lambda')[[h]]$ and $V(\Lambda'')[[h]]$ is

$$(V(\Lambda') \otimes V(\Lambda'')[[h]] \cong \oplus_{\Lambda \in P^+} n^{\Lambda}_{\Lambda',\Lambda''} V(\Lambda)[[h]]. \tag{2.6.13}$$

This is important; it means that the classical result. Theorem 1.12.2, translates immediately to a theorem about tensor product decomposition of $U_h(\mathfrak{g})$-modules.

Let us also record here the following useful fact:

THEOREM 2.6.14. *A basis may be chosen for $V[[h]]$ such that the matrices $\rho^{V[[h]]}(X_i)$, $\rho^{V[[h]]}(Y_i)$ and $\rho^{V[[h]]}(H_i)$ of the indecomposable representation, $(\tilde{\rho}^{V[[h]]} \circ \tilde{\phi}, V[[h]])$, of the Drinfeld-Jimbo QUEAs corresponding to the first fundamental representation, V, of $U(\mathfrak{g})$ are the same matrices as in the classical case.*

2.7. The necessity of quasi-Hopf algebras

The QUEA $U_h(\mathfrak{g})$ is a formal deformation of the classical enveloping algebra $U(\mathfrak{g})$. Theorem 2.4.4 guarantees the existence of the QUEA $U_h(\mathfrak{g})_\phi$, a formal deformation of $U(\mathfrak{g})$ with a trivial algebra structure. In this section we will see that whenever we have a QUEA $U_h(\mathfrak{g})$ we must also have an associated 'quasi-Hopf' structure built on $U(\mathfrak{g})[[h]]$.

Take any pair of finite representations, $(\tilde{\rho}^{V[[h]]}, V[[h]])$ and $(\tilde{\rho}^{W[[h]]}, W[[h]])$ of $U_h(\mathfrak{g})_\phi$. We may assume that $\rho_n^V = \rho_n^W = 0$ for all $n > 0$. The tensor product representation $(\tilde{\rho}^{(V\otimes W)[[h]]}, (V \otimes W)[[h]])$ is defined for all $x \in U(\mathfrak{g})$ by

$$\tilde{\rho}^{(V\otimes W)[[h]]}(x) = (\rho_0^V \otimes \rho_0^W)(\tilde{\Delta}_\phi(x)), \tag{2.7.1}$$

extended $\mathbb{C}[[h]]$-linearly to the whole of $U_h(\mathfrak{g})_\phi$. But this must be equivalent to the representation $(\rho_0^{V\otimes W}, (V \otimes W)[h]])$ of $U_h(\mathfrak{g})_\phi$ defined for all $x \in U(\mathfrak{g})$ by

$$\rho_0^{V\otimes W}(x) = (\rho_0^V \otimes \rho_0^W)(\Delta(x)), \tag{2.7.2}$$

where Δ is the classical coproduct of $U(\mathfrak{g})$ extended $\mathbb{C}[[h]]$-linearly to the whole of $U_h(\mathfrak{g})_\phi$. Therefore there must exist an invertible intertwiner $F_{V,W} : (V \otimes W)[[h]] \to (V \otimes W)[[h]]$ such that for all $x \in U(\mathfrak{g})$

$$(\rho_0^V \otimes \rho_0^W)(\tilde{\Delta}_\phi(x)) = F_{V,W} \circ \left[(\rho_0^V \otimes \rho_0^W)(\Delta(x))\right] \circ F_{V,W}^{-1}. \tag{2.7.3}$$

Therefore, at least in any pair of indecomposable representations of $U(\mathfrak{g})$, the coproduct $\tilde{\Delta}_\phi$ of $U_h(\mathfrak{g})_\phi$ is related through conjugation by some 'F-matrix' to the $\mathbb{C}[[h]]$-linearly extended classical coproduct Δ.

This observation suggests that there may be some element $\mathcal{F} \in (U(\mathfrak{g}) \otimes U(\mathfrak{g}))[[h]]$ such that $\tilde{\Delta}_\phi(x) = \mathcal{F}\Delta(x)\mathcal{F}^{-1}$ for all $x \in U(\mathfrak{g})$. In fact this is the case. It can be shown to follow from the vanishing of $H^1(\mathfrak{g}, U(\mathfrak{g}) \otimes U(\mathfrak{g}))$ that such an $\mathcal{F} \in (U(\mathfrak{g}) \otimes U(\mathfrak{g}))[[h]]$ exists and moreover it may be taken to be 'counital' in the sense that $(\epsilon \otimes \mathrm{id})(\mathcal{F}) = (\mathrm{id} \otimes \epsilon)(\mathcal{F}) = 1$. The observations of the previous paragraph then follow immediately on evaluating in any pair of representations.

We will assume that $U_h(\mathfrak{g})$, and so $U_h(\mathfrak{g})_\phi$, is quasitriangular. Then

$$\tilde{\Delta}_\phi^{\mathrm{op}}(x) = \mathcal{R}^\phi \tilde{\Delta}_\phi(x) \mathcal{R}^{\phi^{-1}}, \tag{2.7.4}$$

from which it follows easily that

$$\Delta^{\mathrm{op}}(x) = \left(\mathcal{F}_{21}^{-1}\mathcal{R}^\phi\mathcal{F}\right) \Delta(x) \left(\mathcal{F}^{-1}\mathcal{R}^{\phi^{-1}}\mathcal{F}_{21}\right), \tag{2.7.5}$$

where $\mathcal{F}_{21} = \tau(\mathcal{F})$ and $\tau : (U(\mathfrak{g}) \otimes U(\mathfrak{g}))[[h]] \to (U(\mathfrak{g}) \otimes U(\mathfrak{g}))[[h]]$ is the usual flip map $\mathbb{C}[[h]]$-linearly extended. In fact since Δ is cocommutative we know that $\Delta^{\mathrm{op}} = \Delta$ and so

$$\left[\Delta(x), \mathcal{R}^{\phi,F}\right] = 0, \tag{2.7.6}$$

where we have introduced $\mathcal{R}^{\phi,F} = \mathcal{F}_{21}^{-1}\mathcal{R}^\phi\mathcal{F}$.

Now consider $(\tilde{\Delta}_\phi \hat{\otimes}\, \mathrm{id}) \circ \tilde{\Delta}_\phi(x)$,

$$(\tilde{\Delta}_\phi \hat{\otimes}\, \mathrm{id}) \circ \tilde{\Delta}_\phi(x) = (\mathcal{F} \otimes 1) \left(\Delta \otimes \mathrm{id}) \circ \tilde{\Delta}_\phi(x)\right) (\mathcal{F}^{-1} \otimes 1), \tag{2.7.7}$$

$$= (\mathcal{F} \otimes 1)((\Delta \otimes \mathrm{id})(\mathcal{F}))\,((\Delta \otimes \mathrm{id}) \circ \Delta(x))\,((\Delta \otimes \mathrm{id})(\mathcal{F}^{-1}))(\mathcal{F}^{-1} \otimes 1). \tag{2.7.8}$$

Similarly we have

$$(\mathrm{id}\,\hat{\otimes} \tilde{\Delta}_\phi) \circ \tilde{\Delta}_\phi(x) = (1 \otimes \mathcal{F})((\mathrm{id} \otimes \Delta)(\mathcal{F}))\,((\mathrm{id} \otimes \Delta) \circ \Delta(x))\,((\mathrm{id} \otimes \Delta)(\mathcal{F}^{-1}))(1 \otimes \mathcal{F}^{-1}) \tag{2.7.9}$$

But by the coassociativity of $U_h(\mathfrak{g})_\phi$ we must then have

$$(\mathrm{id} \otimes \Delta) \circ \Delta(x) = \Phi\,((\Delta \otimes \mathrm{id}) \circ \Delta(x)))\,\Phi^{-1}, \tag{2.7.10}$$

where $\Phi \in (U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes U(\mathfrak{g}))[[h]]$ is given in terms of the 'F-matrices' as

$$\Phi = ((\mathrm{id} \otimes \Delta)(\mathcal{F}^{-1}))(1 \otimes \mathcal{F}^{-1})(\mathcal{F} \otimes 1)((\Delta \otimes \mathrm{id})(\mathcal{F})). \tag{2.7.11}$$

Actually we know that Δ is coassociative so

$$[(\mathrm{id} \otimes \Delta) \circ \Delta(x), \Phi] = 0, \tag{2.7.12}$$

for all $x \in U(\mathfrak{g})$. Notice that it follows from the counital property of $\mathcal{F}$ that $(\mathrm{id} \otimes \epsilon\, \mathrm{id})(\Phi) = 1$.

If $U[[h]]$, $V[[h]]$ and $W[[h]]$ are any three $U(\mathfrak{g})[[h]]$-modules and we denote by $\Phi_{U,V,W}$ the evaluation of Φ on $(U \otimes V \otimes W)[[h]]$, then $\Phi_{U,V,W}$ is an intertwiner between the $U(\mathfrak{g})$-modules $((U \otimes V) \otimes W)[[h]]$ and $(U \otimes (V \otimes W))[[h]]$. Considering a fourth $U(\mathfrak{g})[[h]]$-module $X[[h]]$, we can construct from Φ two different intertwiners between the $U(\mathfrak{g})[[h]]$-modules $(((U \otimes V) \otimes W) \otimes X)[[h]]$ and $(U \otimes (V \otimes (W \otimes X)))[[h]]$. First of all, $((\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi))_{U,V,W,X}$ intertwines between $(((U \otimes V) \otimes W) \otimes X)[[h]]$ and $((U \otimes V) \otimes (W \otimes X))[[h]]$ and then $((\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi))_{U,V,W,X}$ takes us from $((U \otimes V) \otimes (W \otimes X))[[h]]$ to $(U \otimes (V \otimes (W \otimes X)))[[h]]$ so the composition

$$((\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi))_{U,V,W,X} \circ ((\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi))_{U,V,W,X} \tag{2.7.13}$$

is an intertwiner between $(((U \otimes V) \otimes W) \otimes X)[[h]]$ and $(U \otimes (V \otimes (W \otimes X)))[[h]]$. Second we have the composition of intertwiners

$$(1 \otimes \Phi)_{U,V,W,X} \circ ((\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi))_{U,V,W,X} \circ (\Phi \otimes 1)_{U,V,W,X}, \tag{2.7.14}$$

which takes us from $(((U \otimes V) \otimes W) \otimes X)[[h]]$ to $((U \otimes (V \otimes W)) \otimes X)[[h]]$ to $(U \otimes ((V \otimes W) \otimes X))[[h]]$ to $(U \otimes (V \otimes (W \otimes X)))[[h]]$. It would be rather desirable for these two intertwiners to act identically and this is the case. Indeed it is a simple matter to check that

$$((\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi))((\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi)) = (1 \otimes \Phi)((\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi))(\Phi \otimes 1) \tag{2.7.15}$$

from which the desired property of the intertwiners follows immediately.

Some further calculations based on considering what become of the relations

$$(\tilde{\Delta}_\phi \hat{\otimes}\, \mathrm{id}) \circ \tilde{\Delta}_\phi(\mathcal{R}^\phi) = \mathcal{R}^\phi_{13}\mathcal{R}^\phi_{23}, \tag{2.7.16}$$

$$(\mathrm{id}\,\hat{\otimes} \tilde{\Delta}_\phi) \circ \tilde{\Delta}_\phi(\mathcal{R}^\phi) = \mathcal{R}^\phi_{13}\mathcal{R}^\phi_{12}, \tag{2.7.17}$$

lead naturally to the following relations

$$(\Delta \otimes \mathrm{id})(\mathcal{R}^{\phi,F}) = \Phi_{312}\mathcal{R}_{13}^{\phi,F}\Phi_{132}^{-1}\mathcal{R}_{23}^{\phi,F}\Phi, \tag{2.7.18}$$

$$(\mathrm{id} \otimes \Delta)(\mathcal{R}^{\phi,F}) = \Phi_{231}^{-1}\mathcal{R}_{13}^{\phi,F}\Phi_{213}\mathcal{R}_{12}^{\phi,F}\Phi^{-1}. \tag{2.7.19}$$

Thus we see that associated with any QUEA, classical cohomolgy theory demands the existence of a new structure $(U(\mathfrak{g})[[h]], m, \eta, \Delta, \epsilon, S, \Phi, \mathcal{R})$ defined on $U(\mathfrak{g})[[h]]$ with the classical Hopf algebra maps and related to $U_h(\mathfrak{g})_\phi$ through 'twisting' by an element $\mathcal{F} \in (U(\mathfrak{g}) \otimes U(\mathfrak{g}))[[h]]$. This is an example of a quasi-Hopf algebra.

2.8. Quantised coordinate rings

To define quantised coordinate rings $\mathbb{C}_h[G]$ for the classical complex linear algebraic groups G where G is $SL_{l+1}(\mathbb{C})$, $SO_{2l+1}(\mathbb{C})$, $Sp_{2l}(\mathbb{C})$ or $SO_{2l}(\mathbb{C})$ we take the classical case as the paradigm. Recall that there we found that $\mathbb{C}[G] \cong U(\mathfrak{g})^{(\circ)}$ where $U(\mathfrak{g})^{(\circ)}$ is the Hopf algebra generated by the matrix elements of the the defining representation, V, of $\mathfrak{g}$. We have seen that the bijective correspondence between $\mathrm{Rep}(U_h(\mathfrak{g}))$ and $\mathrm{Rep}(U(\mathfrak{g}))$ preserves tensor products, duals, direct sums, submodules and indecomposability. It follows that the classical tensor product decompositions are preserved and in particular the representations which can be obtained within iterated tensor products of the first fundamental representation are the same modulo the correspondence $V \mapsto V[[h]]$. Furthermore this collection of representations will then be closed under the formation of duals since this is true classically. Therefore we should define what we mean by matrix coefficients for representations of formal deformations of Hopf algebras and then define quantised coordinate rings as the Hopf algebras topologically generated by the matrix elements of the representations $(\tilde{\rho}^{V[[h]]}, V[[h]])$ corresponding to the classical defining representations.

If A_h is a formal deformation of a Hopf algebra A, then the $\mathbb{C}[[h]]$-linear dual of A_h is

$$A_h^* = \mathrm{Hom}_{\mathbb{C}[[h]]}(A[[h]], \mathbb{C}[[h]]) = \mathrm{Hom}(A, \mathbb{C})[[h]] = A^*[[h]]. \tag{2.8.1}$$

DEFINITION 2.8.1. Suppose $(\tilde{\rho}^{V[[h]]}, V[[h]])$ is a representation of A_h then the *matrix coefficients* of $(\tilde{\rho}^{V[[h]]}, V[[h]])$ are the elements $\tilde{\rho}_{\tilde{\alpha},\tilde{v}}^{V[[h]]} \in A_h^*$ for all $\tilde{\alpha} \in V^*[[h]]$ and $\tilde{v} \in V[[h]]$ defined by

$$\tilde{\rho}_{\tilde{\alpha},\tilde{v}}^{V[[h]]}(\tilde{a}) = \sum_{n\geq 0}\left(\sum_{p+q+r+s=n}\langle \alpha_p, \rho_q^V(a_r)v_s\rangle\right)h^n. \tag{2.8.2}$$

The *Hopf dual* of A_h, is then defined as the closure in A_h^*, $\widehat{A_h^\circ}$, of the set A_h° of matrix coefficients of all finite rank representations of A_h. The $\mathbb{C}[[h]]$-linear space of matrix coefficients of a representation $(\tilde{\rho}^{V[[h]]}, V[[h]])$ is $\mathbb{C}[[h]]$-linearly spanned by the *matrix elements* $\rho_{ij}^{V[[h]]} = \rho_{\alpha_i,v_j}^{V[[h]]}$.

We are interested in $\widehat{U_h(\mathfrak{g})^\circ}$. Any element $\tilde{\rho}_{\tilde{\alpha},\tilde{v}}^{V[[h]]} \in \widehat{U_h(\mathfrak{g})^\circ}$ is specified by the elements $(\tilde{\rho}_{\tilde{\alpha},\tilde{v}}^{V[[h]]})_n \in U_h(\mathfrak{g})^*$ defined by

$$(\tilde{\rho}_{\tilde{\alpha},\tilde{v}}^{V[[h]]})_n = \sum_{p+q+r=n}\langle \alpha_p, \rho_q^V(\cdot)v_r\rangle. \tag{2.8.3}$$

In particular

$$(\tilde{\rho}^{V[[h]]}_{\tilde{\alpha},\tilde{v}})_0 = \langle \alpha_0, \rho^V_0(\cdot)v_0\rangle \tag{2.8.4}$$

is clearly an element of $U(\mathfrak{g})^\circ$. We know that $U_h(\mathfrak{g})$ can be identified with $U(\mathfrak{g})[[h]]$ as $\mathbb{C}[[h]]$-modules and since $\widehat{U_h(\mathfrak{g})^\circ}$ is defined as the closure in a topologically free module, it must itself be topologically free. Then the bijection between $\mathrm{Rep}(U_h(\mathfrak{g}))$ and $\mathrm{Rep}(U(\mathfrak{g}))$ tells us that $\widehat{U_h(\mathfrak{g})^\circ}$ can be identified with $U(\mathfrak{g})^\circ[[h]]$ as $\mathbb{C}[[h]]$-modules. We have a natural topological Hopf algebra structure on the Hopf dual of $U_h(\mathfrak{g})$ analogous to the Hopf algebra structure on $U(\mathfrak{g})^\circ$.

THEOREM 2.8.2. *$\widehat{U_h(\mathfrak{g})^\circ}$ is a topological Hopf algebra. Indeed, for all $\tilde{v} \in V[[h]]$, $\tilde{w} \in W[[h]]$, $\tilde{\alpha} \in V^*[[h]]$ and $\tilde{\beta} \in W^*[[h]]$, where $(\tilde{\rho}^{V[[h]]}, V[[h]])$ and $(\tilde{\rho}^{W[[h]]}, W[[h]])$ are any pair of representations of $U_h(\mathfrak{g})$ we have*

$$\tilde{\rho}^{V[[h]]}_{\tilde{\alpha},\tilde{v}} + \tilde{\rho}^{W[[h]]}_{\tilde{\beta},\tilde{w}} = \tilde{\rho}^{(V\oplus W)[[h]]}_{\tilde{\alpha}\oplus\tilde{\beta},\tilde{v}\oplus\tilde{w}}, \tag{2.8.5}$$

$$\tilde{\kappa}\tilde{\rho}^{V[[h]]}_{\tilde{\alpha},\tilde{v}} = \tilde{\rho}^{V[[h]]}_{\tilde{\kappa}\tilde{\alpha},\tilde{v}} = \tilde{\rho}^{V[[h]]}_{\tilde{\alpha},\tilde{\kappa}\tilde{v}}, \tag{2.8.6}$$

where $\tilde{\kappa} \in \mathbb{C}[[h]]$, and

$$\tilde{m}(\tilde{\rho}^{V[[h]]}_{\tilde{\alpha},\tilde{v}} \hat{\otimes} \tilde{\rho}^{W[[h]]}_{\tilde{\beta},\tilde{w}}) = \tilde{\rho}^{(V\otimes W)[[h]]}_{\tilde{\alpha}\hat{\otimes}\tilde{\beta},\tilde{v}\hat{\otimes}\tilde{w}}, \tag{2.8.7}$$

$$\eta(1) = \epsilon_{U_h(\mathfrak{g})},$$

$$\tilde{\Delta}(\tilde{\rho}^{V[[h]]}_{\tilde{\alpha},\tilde{v}}) = \sum_i \tilde{\rho}^{V[[h]]}_{\tilde{\alpha},v_i} \hat{\otimes} \tilde{\rho}^{V[[h]]}_{\alpha_i,\tilde{v}}, \tag{2.8.8}$$

$$\epsilon(\tilde{\rho}^{V[[h]]}_{\tilde{\alpha},\tilde{v}}) = \tilde{\alpha}(\tilde{v}), \tag{2.8.9}$$

$$S(\tilde{\rho}^{V[[h]]}_{\tilde{\alpha},\tilde{v}}) = \tilde{\rho}^{V^*[[h]]}_{\tilde{v},\tilde{\alpha}}, \tag{2.8.10}$$

where $\{v_i\}$ and $\{\alpha_i\}$ are dual bases for V and V^* respectively and $\epsilon_{U_h(\mathfrak{g})}$ is the counit in $U_h(\mathfrak{g})$.

It will be clear that the Hopf dual of $U_h(\mathfrak{g})$ is a formal deformation of $U(\mathfrak{g})^\circ$. Moreover the Hopf duals corresponding to equivalent formal deformations of $U(\mathfrak{g})$ are clearly isomorphic and this isomorphism is the identity modulo h. Therefore we could make a considerable simplification by working with $\widehat{U_h(\mathfrak{g})^\circ_\phi}$ rather than $\widehat{U_h(\mathfrak{g})^\circ}$. Indeed in this case the matrix coefficients are specified by the components

$$(\tilde{\rho}^{V[[h]]}_{\tilde{\alpha},\tilde{v}})_n = \sum_{p+q=n} \langle \alpha_p, \rho^V(\cdot)v_r\rangle \tag{2.8.11}$$

$$= \sum_{p+q=n} \rho^V_{\alpha_p,v_q}, \tag{2.8.12}$$

and so are clearly sums of matrix coefficients of $U(\mathfrak{g})$.

We can now define quantised coordinate rings.

DEFINITION 2.8.3. If G is one of the complex Lie groups $SL_n(\mathbb{C})$, $SO_{2n+1}(\mathbb{C})$, $Sp_{2n}(\mathbb{C})$ or $SO_{2n}(\mathbb{C})$ then the *quantised coordinate ring*, $\mathbb{C}_h[G]$, corresponding to a QUEA $U_h(\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of G is the sub-Hopf algebra of $\widehat{U_h(\mathfrak{g})^\circ}$ topologically generated by the matrix coefficients of the representation $(\tilde{\rho}^{V[[h]]}, V[[h]])$ where (ρ^V, V) is the first fundamental representation of $\mathfrak{g}$.

Proceeding as we did in Chapter 1, Section 1.12.6 we can derive relations in $\mathbb{C}_h[G]$. However, important differences appear in the 'quantum' analogue of that analysis. We have seen that the permutation map $\tau_{V,V} : V \otimes V \to V \otimes V$ is a $U(\mathfrak{g})$-module map. However it is not a $U_h(\mathfrak{g})$-module map since $U_h(\mathfrak{g})$ is not cocommutative. Rather, relation (2.3.1) tells us that $\hat{R}_{V,V} : (V \otimes V)[[h]] \to (V \otimes V)[[h]]$ is a $U_h(\mathfrak{g})$-module map where $\hat{R}_{V,V} = \tau_{V,V} \circ R_{V,V}$, $\tau : (V \otimes V)[[h]] \to (V \otimes V)[[h]]$ being the usual flip map on $V \otimes V$ $\mathbb{C}[[h]]$-linearly extended and $R_{V,V} = (\rho^{V[[h]]} \hat{\otimes} \rho^{V[[h]]})(\mathcal{R})$. From the arguments in Chapter 1, Section 1.11 we see that in *every* case ($\mathfrak{g} = \mathfrak{sl}_{l+1}(\mathbb{C}), \mathfrak{so}_{2l+1}(\mathbb{C}), \mathfrak{sp}_{2l}(\mathbb{C}), \mathfrak{so}_{2l}(\mathbb{C})$) we must have relations of the form

$$\sum_{k,l=1}^{n} \hat{R}_{ij,kl} \rho_{ks}^{V[[h]]} \rho_{lt}^{V[[h]]} = \sum_{k,l=1}^{n} \rho_{ik}^{V[[h]]} \rho_{jl}^{V[[h]]} \hat{R}_{kl,st}, \tag{2.8.13}$$

where we have written $\hat{R}$ for $\hat{R}_{V,V}$ and $\hat{R}(v_i \otimes v_j) = \sum_{a,b=1}^{n} \hat{R}_{ab,ij}(v_a \otimes v_b)$. If we write the generators of $\mathbb{C}_h[G]$ as T_{ij} instead of $\rho_{ij}^{V[[h]]}$ then in a popular compact notation these relations could be written simply as

$$\hat{R}_{12} T_1 T_2 = T_1 T_2 \hat{R}_{12}. \tag{2.8.14}$$

This equation is the starting point of the well-known FRT approach to quantum groups. In this approach we do not need to use an R associated with a universal R-matrix. Any matrix satisfying the *matrix* QYBE $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ gives rise to a *coquasitriangular bialgebra* (see Chapter 3 for the definition) and in certain cases a Hopf algebra.

THEOREM 2.8.4. *Given an $N^2 \times N^2$ matrix R satisfying the matrix QYBE together with N^2 symbols T_{ij}, $i, j = 1, \ldots, N$, then $\mathbb{C}\langle T_{ij}\rangle$ is the free algebra over $\mathbb{C}$ generated by the T_{ij}s and I_R is the two-sided ideal in $\mathbb{C}\langle T_{ij}\rangle$ generated by the elements*

$$\hat{R}_{12} T_1 T_2 - T_1 T_2 \hat{R}_{12} \tag{2.8.15}$$

and we define

$$A(R) = \mathbb{C}\langle T_{ij}\rangle / I_R. \tag{2.8.16}$$

There is a unique bialgebra structure on $A(R)$ such that

$$\Delta(T_{ik}) = \sum_{j=1}^{N} T_{ij} \otimes T_{jk}, \quad \epsilon(T_{ij}) = \delta_{ij}, \tag{2.8.17}$$

for all $i, j = 1, \ldots, N$, extended as algebra homomorphisms to the whole of $A(R)$. $A(R)$ is called the FRT bialgebra.

REMARK 2.8.5. In some cases we can quotient $A(R)$ and obtain a Hopf algebra. This is the case when we take R to be the matrix representation (for the $U_h(\mathfrak{g})$-module $V[[h]]$) of the universal R-matrix of one of the Drinfeld-Jimbo QUEAs. Indeed for $A(R)$ corresponding to the R-matrices of $U_h(\mathfrak{sl}_{l+1}(\mathbb{C}))$, $U_h(\mathfrak{so}_{2l+1}(\mathbb{C}))$, $U_h(\mathfrak{sp}_{2l}(\mathbb{C}))$ and $U_h(\mathfrak{so}_{2l}(\mathbb{C}))$ we can quotient $A(R)$ to obtain the well-known FRT Hopf algebras (as they appear in the paper [84]) which 'quantise' the classical coordinate rings of $SL_n(\mathbb{C})$, $SO_{2n+1}(\mathbb{C})$, $Sp_{2n}(\mathbb{C})$ and $SO_{2n}(\mathbb{C})$ respectively. To be precise the previous theorem should then be given in the h-adic setting with quotients of *closures* of ideals etc. However, the matrix entries of R, although strictly formal, make sense when h is specialised to any non-zero complex number and so it is usual to understand these 'FRT Hopf algebras' as defined over $\mathbb{C}$ rather than $\mathbb{C}[[h]]$. We will denote the FRT quantised coordinate rings by $\mathbb{C}_h[G]_{FRT}$.

Another important difference concerns the construction of the 'quantum' versions of the exterior module $\wedge^2(V) = V \wedge V$. Let us recall the classical situation. The permutation operator $\tau_{V,V} : V \otimes V \to V \otimes V$ is diagonalisable with eigenvalues ± 1 such that $\ker(\tau_{V,V} - \mathrm{id}) = \mathrm{Sym}^2(V)$ and $\ker(\tau_{V,V} + \mathrm{id}) = \wedge^2(V)$. As $\tau_{V,V}$ is a $U(\mathfrak{g})$-module map it follows immediately that both $\mathrm{Sym}^2(V)$ and $\wedge^2(V)$ are submodules of $V \otimes V$. For $\mathfrak{g} = \mathfrak{sl}_{l+1}(\mathbb{C})$ this eigenspace decomposition of $V \otimes V$ coincides precisely with the decomposition of $V \otimes V$ into irreducibles. That is

$$V \otimes V \cong V(2\Lambda_{(1)}) \oplus V(\Lambda_{(2)}), \tag{2.8.18}$$

where $\mathrm{Sym}^2(V) \cong V(2\Lambda_{(1)})$ and $\wedge^2(V) \cong V(\Lambda_{(2)})$. For $\mathfrak{g} = \mathfrak{so}_{2l+1}(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{so}_{2l}(\mathbb{C})$ we do not have such a neat result;

$$V \otimes V \cong V(2\Lambda_{(1)}) \oplus V(\Lambda_{(2)}) \oplus V(0), \tag{2.8.19}$$

where $\wedge^2(V) \cong V(\Lambda_{(2)})$ but $\mathrm{Sym}^2(V) \cong V(2\Lambda_{(1)}) \oplus V(0)$. Similarly for $\mathfrak{g} = \mathfrak{sp}_{2l}(\mathbb{C})$

$$V \otimes V \cong V(2\Lambda_{(1)}) \oplus V(\Lambda_{(2)}) \oplus V(0), \tag{2.8.20}$$

where $\mathrm{Sym}^2(V) \cong V(2\Lambda_{(1)})$ but $\wedge^2(V) \cong V(\Lambda_{(2)}) \oplus V(0)$. Equipped with $\mathrm{Sym}^2(V)$ and $\wedge^2(V)$ we can proceed to construct respectively the exterior and symmetric algebras $\wedge(V)$ and $\mathrm{Sym}(V)$ as the quotients of the tensor algebra $T(V)$ by the ideals generated by $\mathrm{Sym}^2(V)$ and $\wedge^2(V)$ respectively. Both $\mathrm{Sym}(V)$ and $\wedge(V)$ are then graded with each homogeneous component a $U(\mathfrak{g})$-module and we recall that a 'determinant' relation in $U(\mathfrak{g})^{(\circ)}$ is obtained by considering the 1-dimensional space, $\wedge^n(V)$, where $\wedge^m(V)$, $m = 1 \dots n$, is the space of degree m homogeneous elements.

For $U_h(\mathfrak{g})$-modules we have seen that we should consider $\hat{R}_{V,V}$ as a quantum version of $\tau_{V,V}$. Indeed, it can be shown that for each of the cases $\mathfrak{g} = \mathfrak{sl}_{l+1}(\mathbb{C})$, $\mathfrak{so}_{2l+1}(\mathbb{C})$, $\mathfrak{sp}_{2l}(\mathbb{C})$, $\mathfrak{so}_{2l}(\mathbb{C})$, the operator $\hat{R}_{V,V}$ corresponding to the universal R-matrix of the Drinfeld-Jimbo QUEA $U_h(\mathfrak{g})$ is diagonalisable. Moreover, it is a remarkable fact that in each of these cases the eigenspace decomposition of $\hat{R}_{V,V}$ *is precisely the decomposition of* $(V \otimes V)[[h]]$ *into its indecomposable submodules.* In the case of $U_h(\mathfrak{sl}_{l+1}(\mathbb{C}))$, $\hat{R}_{V,V}$ has just two eigenvalues which up to a common factor may be taken to be $e^{h/2}$ and $-e^{-h/2}$. In an obvious notation we then define

$$\mathrm{Sym}_h^2(V[[h]]) = \ker(\hat{R}_{V,V} - e^{h/2}\,\mathrm{id})\widehat{\ }, \quad \wedge_h^2(V[[h]]) = \ker(\hat{R}_{V,V} + e^{-h/2}\,\mathrm{id})\widehat{\ }, \tag{2.8.21}$$

where $\widehat{\ }$ indicates that we take the closure in $(V \otimes V)[[h]]$. In the case of $U_h(\mathfrak{so}_{2l+1}(\mathbb{C}))$ and $U_h(\mathfrak{so}_{2l}(\mathbb{C}))$, $\hat{R}_{V,V}$ has three eigenvalues which up to a common factor may be taken to be $e^{h/2}$, $-e^{-h/2}$ and $e^{(1-n)h/2}$ where $n = 2l + 1$ for $U_h(\mathfrak{so}_{2l+1}(\mathbb{C}))$ and $n = 2l$ for $U_h(\mathfrak{so}_{2l}(\mathbb{C}))$. We then define

$$\mathrm{Sym}_h^2(V[[h]]) = \ker(\hat{R}_{V,V} - e^{h/2}\,\mathrm{id})\widehat{\ } \oplus \ker(\hat{R}_{V,V} - e^{(1-n)h/2}\,\mathrm{id})\widehat{\ }, \tag{2.8.22}$$

$$\wedge_h^2(V[[h]]) = \ker(\hat{R}_{V,V} + e^{-h/2}\,\mathrm{id})\widehat{\ }. \tag{2.8.23}$$

In the case of $U_h(\mathfrak{sp}_{2l}(\mathbb{C}))$, $\hat{R}_{V,V}$ has three eigenvalues which up to a common factor may be taken to be $e^{h/2}$, $-e^{-h/2}$ and $-e^{-(1+2l)h/2}$ and we set

$$\mathrm{Sym}_h^2(V[[h]]) = \ker(\hat{R}_{V,V} - e^{h/2}\,\mathrm{id})\widehat{\ }, \tag{2.8.24}$$

$$\wedge_h^2(V[[h]]) = \ker(\hat{R}_{V,V} + e^{-h/2}\,\mathrm{id})\widehat{\ } \oplus \ker(\hat{R}_{V,V} - e^{-(1+2l)h/2}\,\mathrm{id})\widehat{\ }. \tag{2.8.25}$$

As $\hat{R}_{V,V}$ is a $U_h(\mathfrak{g})$-module map, in each case both $\mathrm{Sym}_h^2(V[[h]])$ and $\wedge_h^2(V[[h]])$ are $U_h(\mathfrak{g})$-submodules of $(V\otimes V)[[h]]$. We then define quantum analogues of the symmetric and exterior algebras, $\wedge_h(V[[h]])$ and $\mathrm{Sym}_h(V[[h]])$ as the quotients of the tensor algebra $T(V)[[h]]$ by the closure of the ideals generated by $\mathrm{Sym}_h^2(V[[h]])$ and $\wedge_h^2(V[[h]])$ respectively. Both $\mathrm{Sym}_h(V)$ and $\wedge_h(V)$ are then graded with each homogeneous component a $U_h(\mathfrak{g})$-module.

Adopting these definitions, it turns out that we are able to recover a quantum analogue of the 'determinant' relation in $\mathbb{C}_h[G]$ since $\wedge_h^n(V)$, the space of degree n homogeneous elements, is 1-dimensional for $n = l+1, 2l+1, 2l, 2l$ in the respective cases $\mathfrak{g} = \mathfrak{sl}_{l+1}(\mathbb{C}), \mathfrak{so}_{2l+1}(\mathbb{C}), \mathfrak{sp}_{2l}(\mathbb{C}), \mathfrak{so}_{2l}(\mathbb{C})$. It can be shown that in the case of $U_h(\mathfrak{sl}_{l+1}(\mathbb{C}))$, $V^*[[h]] \cong \wedge_h^l(V[[h]])$, from which the explicit form of the antipode for $\mathbb{C}_h[SL_n(\mathbb{C})]$ is obtained while for $U_h(\mathfrak{so}_{2l+1}(\mathbb{C}))$, $U(\mathfrak{sp}_{2l}(\mathbb{C}))$ and $U(\mathfrak{so}_{2l}(\mathbb{C}))$ we use $V^*[[h]] \cong V[[h]]$, which is immediate. Furthermore, in a fashion entirely analogous to the classical case we obtain further relations in these cases, namely,

$$T^t\mathbf{X}T\mathbf{X}^{-1} = \mathbf{I} \tag{2.8.26}$$

where T denotes the matrix of generators and $\mathbf{X}$ is the matrix corresponding to the isomorphism map $\phi : V[[h]] \to V^*[[h]]$.

REMARK 2.8.6. As we have already remarked, the FRT bialgebras $A(R)$ can be factored by certain ideals to obtain the familiar FRT Hopf algebras of [**84**]. These ideals are essentially those corresponding to the relations in $\mathbb{C}_h[G]$ which we have just described. In [**84**] these ideals are introduced virtually out of thin air as part of the *definition* of the quantised coordinate rings $\mathbb{C}_h[G]_{FRT}$ in terms of generators and relations. By contrast, in the approach we have described here we find that such relations are indeed *in* $\mathbb{C}_h[G]$ but it is not clear that these are *all* the relations in $\mathbb{C}_h[G]$. Actually, in the case of $\mathbb{C}_h[SL_n(\mathbb{C})]$ Guichardet [**46**] has given an elegant proof that the relations (2.8.14) together with the determinant relation are indeed the only relations in $\mathbb{C}_h[SL_n(\mathbb{C})]$ so demonstrating that the FRT quantisation $\mathbb{C}_h[SL_n(\mathbb{C})]_{FRT}$ and $\mathbb{C}_h[SL_n(\mathbb{C})]$ are isomorphic. In the cases $\mathbb{C}_h[SO_n(\mathbb{C})]$ and $\mathbb{C}_h[Sp_{2n}(\mathbb{C})]$ a proof does not seem to have appeared. In Section 2.10 we prove that the non-standard quantised coordinate ring one would obtain by an FRT approach from the Jordanian R-matrix is isomorphic to that defined as here via the non-standard QUEA.

In Chapter 3 we work with matrices satisfying the QYBE for which there are no *known* universal R-matrices. In these cases the FRT construction is the only means of constructing a quantum group.

EXAMPLE 2.8.7. We consider the example of $\mathbb{C}_h[SL_2(\mathbb{C})]$. Then the relations (2.8.14) are

$$T_{12}T_{11} = e^{h/2}T_{11}T_{12}, \quad T_{22}T_{12} = e^{h/2}T_{12}T_{22}, \tag{2.8.27}$$

$$T_{21}T_{11} = e^{h/2}T_{11}T_{21}, \quad T_{22}T_{21} = e^{h/2}T_{21}T_{22}, \tag{2.8.28}$$

$$T_{21}T_{12} = T_{12}T_{21}, \quad T_{22}T_{11} - T_{11}T_{22} = (e^{h/2} - e^{-h/2})T_{12}T_{21}. \tag{2.8.29}$$

Denoting by Φ the isomorphism map between $\wedge^2(V[[h]])$ and $V(0)[[h]]$, we have

$$\begin{aligned}\Phi(x \triangleright (v_0 \wedge_h v_1)) &= \Phi(x \triangleright (v_0 \otimes v_1 - e^{-h/2} v_1 \otimes v_0)) \\ &= \Phi((\rho_{00}^{V[[h]]} \rho_{11}^{V[[h]]} - e^{-h/2} \rho_{01}^{V[[h]]} \rho_{10}^{V[[h]]})(x) v_0 \otimes v_1 \\ &- (e^{-h/2} \rho_{11}^{V[[h]]} \rho_{00}^{V[[h]]} - \rho_{10}^{V[[h]]} \rho_{01}^{V[[h]]})(x) v_1 \otimes v_0) \\ &= (\rho_{11}^{V[[h]]} \rho_{00}^{V[[h]]} - e^{h/2} \rho_{01}^{V[[h]]} \rho_{10}^{V[[h]]})(x) \Phi(v_0 \wedge_h v_1) \\ &= \kappa(\rho_{11}^{V[[h]]} \rho_{00}^{V[[h]]} - e^{h/2} \rho_{01}^{V[[h]]} \rho_{10}^{V[[h]]})(x) v_0^0, \end{aligned} \tag{2.8.30}$$

and

$$x \triangleright \Phi(v_0 \wedge v_1) = \kappa \mathbf{1}(x), \tag{2.8.31}$$

for all $x \in U_h(\mathfrak{sl}_2(\mathbb{C}))$ from which we deduce the 'determinant' relation

$$T_{22}T_{11} - e^{h/2} T_{12} T_{21} = \mathbf{1}. \tag{2.8.32}$$

2.9. The Diamond Lemma

We recall here a very useful result for establishing a basis for an algebra presented in terms of generators and relations. It is due to Bergman [**11**] and called the Diamond Lemma.

Given a set of symbols X we denote by $k\langle X\rangle$ the algebra over a commutative ring k freely generated by the symbols. A *reduction system* for X is a family $S_i = (z_i, \theta_i)_{i \in I}$ where $z_i \in \langle X\rangle$, the set of monomials (a unital monoid), and $\theta_i \in k\langle X\rangle$. Then for any $a, b \in \langle X\rangle$ and any $i \in I$ a *reduction* is a k-linear map $r_{a,i,b} : k\langle X\rangle \to k\langle X\rangle$ such that $r_{a,i,b}(az_ib) = a\theta_i b$ but $r_{a,i,b}$ leaves all other elements of $\langle X\rangle$ fixed. A monomial is called *irreducible* if it is not of the form az_ib for any $a, b \in \langle X\rangle$ and $i \in I$. We denote by J_S the two-sided ideal in $k\langle X\rangle$ generated by the elements $z_i - \theta_i$ for all $i \in I$.

An *overlap ambiguity* is a tuple (i, j, a, b, c) with $i, j \in I$ and $a, b, c \in \langle X\rangle - 1$ such that $z_i = ab$ and $z_j = bc$. It is *resolvable* if there exist products of reductions r and r' such that $r(z_ic) = r'(az_i)$.

An *inclusion ambiguity* is a tuple (i, j, a, b, c) with $i \neq j \in I$ and $a, b, c \in \langle X\rangle$ such that $z_i = b$ and $z_j = abc$. It is *resolvable* if there exist products of reductions r and r' such that $r(az_ib) = r'(t_j)$.

A partial ordering on $\langle X\rangle$ is called *monoidal* if for any $a, b, b', c \in \langle X\rangle$ $b \leq b' \implies abc \leq ab'c$. It is said to be compatible with the reduction system S_i if for all $i \in I$ z_i is a linear combination of monomials strictly less than z_i. The ordering satisfies the *descending chain condition* if all decreasing sequences of elements of $\langle X\rangle$ terminate.

THEOREM 2.9.1. *If every ambiguity of S_i is resolvable and there is a monoidal partial ordering on $\langle X\rangle$ compatible with S_i then the irreducible monomials of $\langle X\rangle$ form a basis of $k\langle X\rangle/J_S$. Moreover given any element $A \in k\langle X\rangle$, every sequence of elements (A_n) obtained by successive aplications of reductions terminates and the final element is independent of the combination of reductions applied.*

In many situations, in particular those of Chapter 4, the correct ordering on $\langle X\rangle$ is easily set up in the following manner: If $X = \{x_1, \ldots, x_n\}$ then we order X according to

$x_i \leq x_j$ if $i \leq j$ and extend this order to $\langle X \rangle$ by ordering monomials in the first place according to their total length and then lexicographically according to the ordering on X. For example, this ordering allows us to prove the Poincaré-Birkhoff-Witt theorem for classical Lie algebras in a matter of a few lines.

In quantum group applications another ordering is often required: If $X = \{x_1, \dots, x_n\}$ then we order X according to $x_i \leq x_j$ if $i \leq j$. We then order $\langle X \rangle$ first of all according to the total number of occurences of elements of some subset of X, secondly by the total length and then finally lexicographically with respect to the order on X.

Our quantum groups appear as topological algebras $A_h = (\mathbb{C}\langle X \rangle)[[h]]/\widehat{K}$, where $\widehat{K}$ is the closure of the two-sided ideal in $(\mathbb{C}\langle X \rangle)[[h]]$ generated by a set $\{z_i - \tilde{t}_i\}_{i \in I}$ where $z_i \in \langle X \rangle$ and $\tilde{t}_i \in (\mathbb{C}\langle X \rangle)[[h]]$. We want to apply the Diamond lemma to such algebras and obtain bases but we can see that the reduction systems we want to specify do not belong to $\langle X \rangle \times \mathbb{C}[[h]]\langle X \rangle$. Now $(\mathbb{C}\langle X \rangle)[[h]]$ is the inverse limit of the algebras $(\mathbb{C}\langle X \rangle)[[h]]/h^n(\mathbb{C}\langle X \rangle)[[h]]$ with $\pi_n : (\mathbb{C}\langle X \rangle)[[h]] \to (\mathbb{C}\langle X \rangle)[[h]]/h^n(\mathbb{C}\langle X \rangle)[[h]]$ the canonical map. We recall from Theorem 2.2.4 that in these circumstances we have

$$(\mathbb{C}\langle X \rangle)[[h]]/\widehat{K} = \varprojlim((\mathbb{C}\langle X \rangle)[[h]]/h^n(\mathbb{C}\langle X \rangle)[[h]])/\pi_n(\widehat{K}). \tag{2.9.1}$$

The idea is to apply the Diamond lemma to the algebras

$$((\mathbb{C}\langle X \rangle)[[h]]/h^n(\mathbb{C}\langle X \rangle)[[h]])/\pi_n(\widehat{K}), \tag{2.9.2}$$

which are algebras over the commutative ring $\mathbb{C}[[h]]/h^n\mathbb{C}[[h]]$ generated by X and the relations $z_i = \pi_n(\tilde{t}_i)$. Indeed the reduction systems we should consider for these algebras are obviously

$$(z_i, \tilde{t}_i \bmod h^n(\mathbb{C}\langle X \rangle)[[h]]). \tag{2.9.3}$$

Then, *if the conditions of the diamond lemma are satisfied the topological basis of A_h is provided by the monomials which do not contain any z_is.*

We can identify

$$((\mathbb{C}\langle X \rangle)[[h]]/h(\mathbb{C}\langle X \rangle)[[h]])/\pi_1(\widehat{K}) \tag{2.9.4}$$

with the $\mathbb{C}$-algebra $A_0 = \mathbb{C}\langle X \rangle/K_0$ where K_0 is the two-sided ideal in $\mathbb{C}\langle X \rangle$ generated by $\{z_i - t_{i,0}\}_{i \in I}$. Typically A_h is the quantum algebra and A_0 is the corresponding classical algebra. Then if the conditions of the diamond lemma are satisfied for each

$$((\mathbb{C}\langle X \rangle)[[h]]/h^n(\mathbb{C}\langle X \rangle)[[h]])/\pi_n(\widehat{K}), \tag{2.9.5}$$

they are in particular satisfied for A_0 and we see that $A_0[[h]]$ and A_h can be identified as $\mathbb{C}[[h]]$-modules.

2.10. Application to the Jordanian quantum groups

In this section we will apply the techniques described in the previous section to prove two results concerning the non-standard Jordanian quantum groups.

We recall the Jordanian QUEA is defined as the topological algebra $U_h(\mathfrak{sl}_2(\mathbb{C})) = A[[h]]/\widehat{K}$ where $A = \mathbb{C}\langle X, Y, H \rangle$ is the algebra of non-commutative polynomials in the generators X, Y and H over $\mathbb{C}$, and $\widehat{K}$ is the closure in $A[[h]]$ of the two-sided ideal generated by

$$[H,X]-2\frac{\sinh hX}{h}, \quad [X,Y]-H, \tag{2.10.1}$$

$$[H,Y]+2Y(\cosh hX)+hH(\sinh hX)-\frac{1}{2}h(\sinh 2hX). \tag{2.10.2}$$

REMARK 2.10.1. The usual description of $\widehat{K}$ is as the ideal generated by

$$[H,X]-2\frac{\sinh hX}{h}, \quad [H,Y]+Y(\cosh hX)+(\cosh hX)Y, \quad [X,Y]-H, \tag{2.10.3}$$

but it is straightforward to check that this is equivalent to $\widehat{K}$.

Note that $\frac{\sinh hX}{h} = X \bmod h$ is well defined and that $U_h(\mathfrak{sl}_2(\mathbb{C})) = U(\mathfrak{sl}_2(\mathbb{C})) \bmod h$. We must consider the reduction system in $A[[h]]/h^nA[[h]]$ specified by

$$(XH, HX-\frac{2}{h}P_1(X)), \tag{2.10.4}$$

$$(HY, YH-2YP_2(X)-hHP_1(X)+\frac{1}{2}hP_1(2X)), \tag{2.10.5}$$

$$(XY, YX+H), \tag{2.10.6}$$

where $P_1(X) = (\sinh hX) \bmod h^nA[[h]]$ and $P_2 = (\cosh hX) \bmod h^nA[[h]]$ may be regarded as polynomials in X with coefficients in $C[[h]]/h^nC[[h]]$. First of all we need to check that all ambiguities are resolvable. There is only one, XHY, but we need the following lemma whose proof is straightforward.

LEMMA 1. *In $A[[h]]/h^nA[[h]]$*

$$2P_1(X)^2 = P_2(2X)-1 \tag{2.10.7}$$

and the reduction system appled to X^mY yields

$$X^mY = YX^m + mHX^{m-1} - \frac{m(m-1)}{h}X^{m-2}P_1(X), \tag{2.10.8}$$

from which it follows that the reduction system applied to $P_1(X)Y$ yields

$$P_1(X)Y = YP_1(X)+hHP_2(X)-hP_1(X)^2. \tag{2.10.9}$$

Consider $(XH)Y$:

$$\begin{aligned}(XH)Y &= (HX-\frac{2}{h}P_1(X))Y \\ &= HYX+H^2-\frac{2}{h}YP_1(X)-2HP_2(X)+2P_1(X)^2 \\ &= YHX-2YXP_2(X)-hHXP_1(X)+\frac{1}{2}hXP_1(2X) \\ &+H^2-\frac{2}{h}YP_1(X)-2HP_2(X)+2P_1(X)^2,\end{aligned} \tag{2.10.10}$$

while

$$\begin{aligned}X(HY) &= XYH - 2XYP_2(X) - hXHP_1(X) + \frac{1}{2}hXP_1(2X)) \\ &= YXH + H^2 - 2YXP_2(X) - 2HP_2(X) - hHXP_1(X) + 2P_1(X)^2 + \frac{1}{2}hXP_1(2X) \\ &= YHX - \frac{2}{h}YP_1(X) + H^2 - 2YXP_2(X) - 2HP_2(X) - hHXP_1(X) \\ &+ \frac{1}{2}hXP_1(2X) + 2P_1(X)^2, \end{aligned} \tag{2.10.11}$$

Thus the ambiguity is resolvable. We order the set $\{Y, H, X\}$ according to $Y < H < X$ and order monomials of $A[[h]]/h^n A[[h]]$ first of all by the total number of occurences of H and Y, then by the total length of the monomials and finally lexicographically. This ordering is clearly compatible with the reduction system and so we have proved that a topological basis for $U_h(\mathfrak{sl}_2(\mathbb{C}))$ is provided by the monomials $\{Y^\alpha H^\beta X^\gamma : \alpha, \beta, \gamma \in \mathbb{Z}_{\geq 0}\}$. In particular, we can identify $U_h(\mathfrak{sl}_2(\mathbb{C})) = U(\mathfrak{sl}_2(\mathbb{C}))[[h]]$ as $\mathbb{C}[[h]]$-modules and have proved that $U_h(\mathfrak{sl}_2(\mathbb{C}))$ is a formal deformation of $U(\mathfrak{sl}_2(\mathbb{C}))$.

In fact $U_h(\mathfrak{sl}_2(\mathbb{C}))$ is isomorphic as an *algebra* to a formal deformation of $U(\mathfrak{g})$ with the trivial algebra structure [**1**]. Defining

$$Z_+ = \frac{2}{h}(\tanh \frac{hX}{2}), \tag{2.10.12}$$

$$Z_- = (\cosh \frac{hX}{2})Y(\cosh \frac{hX}{2}), \tag{2.10.13}$$

$$K = H, \tag{2.10.14}$$

then we have

$$[K, Z_\pm] = \pm 2Z_\pm, \quad [Z_+, Z_-] = K. \tag{2.10.15}$$

We will now consider $\mathbb{C}_h[SL_2(\mathbb{C})]$. It is topologically generated by the matrix coefficients of the $U_h(\mathfrak{sl}_2(\mathbb{C}))$ representation which corresponds, via the bijection between $\mathrm{Rep}(U_h(\mathfrak{sl}_2(\mathbb{C})))$ and $\mathrm{Rep}(U(\mathfrak{g}))$, to the defining representation of $U(\mathfrak{sl}_2(\mathbb{C}))$. The topological Hopf algebra structure on $U_h(\mathfrak{sl}_2(\mathbb{C}))$ is as follows:

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \tag{2.10.16}$$

$$\Delta(Y) = Y \otimes e^{hX} + e^{-hX} \otimes Y, \tag{2.10.17}$$

$$\Delta(H) = H \otimes e^{hX} + e^{-hX} \otimes H, \tag{2.10.18}$$

$$\epsilon(X) = 0, \quad \epsilon(Y) = 0, \quad \epsilon(H) = 0, \tag{2.10.19}$$

$$S(X) = -X, \quad S(Y) = -e^{hX}Ye^{-hX}, \quad S(H) = -e^{hX}He^{-hX}. \tag{2.10.20}$$

It is not difficult to see that we can choose a topological basis, $\{v_0, v_1\}$ say, for the $U_h(\mathfrak{sl}_2(\mathbb{C}))$-module $V[[h]]$ corresponding to the defining module, V, of $U(\mathfrak{g})$ such that the representation matrices of the quantum generators are the same as those for the corresponding classical generators. Then using the coproduct for $U_h(\mathfrak{sl}_2(\mathbb{C}))$ we obtain the following representation matrices in the tensor product representation $(V \otimes V)[[h]]$

with basis $\{v_0 \otimes v_0, v_0 \otimes v_1, v_1 \otimes v_0, v_1 \otimes v_1\}$:

$$\rho^{(V\otimes V)[[h]]}(H) = \begin{pmatrix} 2 & h & -h & 0 \\ 0 & 0 & 0 & h \\ 0 & 0 & 0 & -h \\ 0 & 0 & 0 & -2 \end{pmatrix} \tag{2.10.21}$$

$$\rho^{(V\otimes V)[[h]]}(X) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.10.22}$$

$$\rho^{(V\otimes V)[[h]]}(Y) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -h & 0 \\ 1 & h & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \tag{2.10.23}$$

The operator $\hat{R}_{V,V} : (V \otimes V)[[h]] \to (V \otimes V)[[h]]$ defined in this basis by

$$\hat{R}_{V,V} = \begin{pmatrix} 1 & h & -h & h^2 \\ 0 & 0 & 1 & -h \\ 0 & 1 & 0 & h \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.10.24}$$

is readily checked to commute with the tensor product representations of X, Y and H. Thus $\hat{R}_{V,V}$ is an intertwiner from $(V\otimes V)[[h]] \to (V\otimes V)[[h]]$. $\hat{R}_{V,V}$ is diagonalisable with eigenvalues ± 1 so we define

$$\begin{aligned} \mathrm{Sym}^2_h(V[[h]]) &= \ker(\hat{R}_{V,V} - \mathrm{id})\widehat{\ } && (2.10.25)\\ &= \mathbb{C}[[h]] \cdot (v_0 \otimes v_0 - hv_1 \otimes v_0) \oplus \mathbb{C}[[h]] \cdot (v_0 \otimes v_1 + v_1 \otimes v_0) && (2.10.26)\\ &\oplus \mathbb{C}[[h]] \cdot v_1 \otimes v_1, && (2.10.27) \end{aligned}$$

and

$$\begin{aligned} \wedge^2_h(V[[h]]) &= \ker(\hat{R}_{V,V} + \mathrm{id})\widehat{\ } && (2.10.28)\\ &= \mathbb{C}[[h]] \cdot (v_0 \otimes v_1 - v_1 \otimes v_0 + hv_1 \otimes v_1). && (2.10.29) \end{aligned}$$

These are submodules of $(V\otimes V)[[h]]$ isomorphic to $V(2\Lambda_{(1)})[[h]]$ and $V(0)[[h]]$ respectively. Writing $a = \tilde{\rho}^{V[[h]]}_{0,0}$, $b = \tilde{\rho}^{V[[h]]}_{0,1}$, $c = \tilde{\rho}^{V[[h]]}_{1,0}$, $d = \tilde{\rho}^{V[[h]]}_{1,1}$ and following the general procedure outlined in Section 2.8 we obtain the following relations in $\mathbb{C}_h[SL_2(\mathbb{C})]$:

$$\begin{gathered} ca = ac + hc^2, \quad cd = dc + hc^2, \\ db = bd - h(ad - bc - hac - d^2), \\ ab = ba - h(ad - bc - hac - a^2), \\ cb = bc + hac + hdc + h^2c^2, \quad da = ad - hac + hdc. \end{gathered} \tag{2.10.30}$$

The isomorphism between $\wedge^2_h(V[[h]])$ and $V(0)[[h]]$ quickly leads to the determinant relation

$$ad - bc - hac = 1. \tag{2.10.31}$$

Now define $\mathbb{C}_h[SL_2(\mathbb{C})]_{FRT} = P[[h]]/\widehat{I}$ where $P = \mathbb{C}\langle T_{11}, T_{12}, T_{21}, T_{22}\rangle$ and $\widehat{I}$ is the closure in $P[[h]]$ of the two-sided ideal generated by

$$\begin{gathered} T_{21}T_{11} - T_{11}T_{21} - hT_{21}^2, \quad T_{21}T_{22} - T_{22}T_{21} - hT_{21}^2, \\ T_{22}T_{12} - T_{12}T_{22} + h(1 - T_{22}^2), \\ T_{11}T_{12} - T_{12}T_{11} + h(1 - T_{11}^2), \\ T_{21}T_{12} - T_{12}T_{21} - hT_{11}T_{21} - hT_{22}T_{21} - h^2T_{21}^2, \quad T_{22}T_{11} - 1 - T_{12}T_{21} - hT_{22}T_{21}, \end{gathered} \tag{2.10.32}$$

$$T_{11}T_{22} - T_{12}T_{21} - hT_{11}T_{21} - 1. \tag{2.10.33}$$

$\mathbb{C}_h[SL_2(\mathbb{C})]_{FRT}$ is the FRT Hopf algebra corresponding to the matrix $R = \tau \circ \hat{R}_{V,V}$ (which is readily verified to satisfy the matrix QYBE). There is obviously an algebra homomorphism $\mathbb{C}_h[SL_2(\mathbb{C})]_{FRT} \to \mathbb{C}_h[SL_2(\mathbb{C})]$ such that $\psi(T_{11}) = a$, $\psi(T_{12}) = b$, $\psi(T_{21}) = c$ and $\psi(T_{22}) = d$. We know that $\mathbb{C}_h[SL_2(\mathbb{C})]$ is a formal deformation of $U(\mathfrak{g})^\circ$ since $U_h(\mathfrak{sl}_2(\mathbb{C}))$ is a formal deformation of $U(\mathfrak{g})$. Furthermore, a straightforward application of the diamond lemma confirms that $\mathbb{C}_h[SL_2(\mathbb{C})]_{FRT}$ may be identified as a $\mathbb{C}[[h]]$-module with $\mathbb{C}_h[SL_2(\mathbb{C})]_{FRT_0}[[h]]$ where $\mathbb{C}_h[SL_2(\mathbb{C})]_{FRT_0} = \mathbb{C}\langle T_{11}, T_{12}, T_{21}, T_{22}\rangle/I_0$ and I_0 is the 'classical' ideal generated by

$$\begin{gathered} T_{21}T_{11} - T_{11}T_{21}, \quad T_{21}T_{22} - T_{22}T_{21}, \\ T_{22}T_{12} - T_{12}T_{22}, \quad T_{11}T_{12} - T_{12}T_{11}, \\ T_{21}T_{12} - T_{12}T_{21}, \quad T_{22}T_{11} - T_{12}T_{21} - 1, \end{gathered} \tag{2.10.34}$$

$$T_{11}T_{22} - T_{12}T_{21} - 1, \tag{2.10.35}$$

so $\mathbb{C}_h[SL_2(\mathbb{C})]_{FRT}$ is topologically free and obviously isomorphic modulo h to $U(\mathfrak{g})^\circ$. But since ψ is a homomorphism between topologically free modules, we know it is of the form $\psi = \sum_{n\geq 0} \psi_n h^n$ and is bijective if and only if ψ_0 is bijective. We have just observed that this is indeed the case so $\mathbb{C}_h[SL_2(\mathbb{C})]_{FRT} \cong \mathbb{C}_h[SL_2(\mathbb{C})]$.

2.11. The rational form of a QUEA

The quantum groups $U_h(\mathfrak{g})$ are algebras over the ring $\mathbb{C}[[h]]$ and whilst expressions such as e^{hX} look familiar enough and indeed have a well defined meaning they are essentially formal objects. We *cannot*, as is certainly tempting, specialise h to any particular value other than 0 and expect to obtain quantities with any mathematical meaning. The notions of convergence for objects such as e^{hX} *only* make sense as long as h remains an indeterminate.

This situation is the motivation behind the construction of 'rational forms' of QUEAs. Here we will just recall the definition for $U_q(\mathfrak{sl}_2(\mathbb{C}))$.

DEFINITION 2.11.1. For $q \in \mathbb{C}$ such that $q \neq 0$ and $q^2 \neq 1$, $U_q(\mathfrak{sl}_2(\mathbb{C}))$ is the algebra over $\mathbb{C}$ generated by the symbols E, F, K, and K^{-1} subject to the relations

$$KK^{-1} = 1 = K^{-1}K, \tag{2.11.1}$$

$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \tag{2.11.2}$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}. \tag{2.11.3}$$

THEOREM 2.11.2. *$U_q(\mathfrak{sl}_2(\mathbb{C}))$ is a Hopf algebra over $\mathbb{C}$ with Hopf maps, Δ, ϵ and S defined on the generators by*

$$\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}, \tag{2.11.4}$$

$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F, \tag{2.11.5}$$

$$\epsilon(E) = \epsilon(F) = 0, \quad \epsilon(K) = \epsilon(K^{-1}) = 1, \tag{2.11.6}$$

$$S(E) = -K^{-1}E, \quad S(F) = -FK, \quad S(K) = K^{-1}, \tag{2.11.7}$$

and extended respectively as algebra, algebra and antialgebra maps.

There is a rational form $U_q(\mathfrak{g})$ corresponding to each $U_h(\mathfrak{g})$ and indeed many authors refer to *these* as *the* quantum groups. However there are important differences between the two objects. Indeed there is no equivalent of the isomorphism ϕ_{DJ} for the Hopf algebras $U_q(\mathfrak{sl}_2(\mathbb{C}))$ and they are not quasitriangular. However, modulo a few technical issues the representation theory of $U_q(\mathfrak{g})$ is still very similar to $U_h(\mathfrak{g})$ and so to $U(\mathfrak{g})$.

2.12. Semiclassical theory — quantum groups at first order

A very important situation which arises in quantum physics is to construct a quantisation of a given classical system. The standard approach is first to describe the classical system in the Hamiltonian formalism where we deal with a manifold whose coordinates are position and momenta variables, and associate real valued functions on the manifold with physical quantities. The equations of motion of these physical quantities are then given in terms of Poisson brackets. Quantising the system amounts to replacing, in the equations of motion, the real valued functions by corresponding self-adjoint operators acting on a Hilbert space, and the Poisson brackets by commutators. The new equations are the equations of motion in the Heisenberg picture of the quantum mechanical system. The radical nature of the passage from commutative functions on phase space to noncommutative operators on Hilbert space has, since the early years of quantum mechanics, prompted a number of authors to attempt a less dramatic 'construction' of quantum mechanics from classical mechanics. Indeed, based on early work of Weyl [**102**] and Moyal [**75**], Bayen et al.[**9**] established that quantum mechanics really could be interpreted as a deformation, in the mathematical sense we have already encountered, of classical mechanics. In this approach, the classical Poisson brackets are recovered at order h in the deformation parameter, Planck's constant. The practical applicability of this approach to quantum mechanics is not clear but it is certainly psychologically significant and provides motivation for investigations relating to quantum groups.

DEFINITION 2.12.1. A *Poisson algebra* A is an associative algebra equipped with a bilinear, skew-symmetric map $\{\ ,\ \} : A \otimes A \to A$, the *Poisson bracket*, such that

$$\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0, \tag{2.12.1}$$

$$\{ab, c\} = a\{b, c\} + \{a, c\}b. \tag{2.12.2}$$

In purely algebraic terms we could understand physical quantisation as the deformation of a commutative Poisson algebra, A_0 over $\mathbb{C}$, resulting in a non-commutative associative algebra A_h over $\mathbb{C}[[h]]$ with $A_0 \cong A_h/hA_h$ and equipped with a commutator $[\ ,\ ] : A_h \hat{\otimes} A_h \to A_h$ such that $\{a_0, b_0\} = (h^{-1}[a_h, b_h]) \bmod h$. With this point of view the Poisson structure appears at first order in the deformation parameter h.

With quantum groups such as $U_h(\mathfrak{g})$ it is natural, in the light of these observations, to look at their first order structures.

EXAMPLE 2.12.2. Let us consider $U_h(\mathfrak{sl}_2(\mathbb{C}))$. We know that, mod$h$, $U_h(\mathfrak{sl}_2(\mathbb{C}))$ degenerates into the cocommutative $U(\mathfrak{sl}_2(\mathbb{C}))$ so that we must have $\Delta_h - \Delta_h^{\mathrm{op}} = 0 \bmod h$ and we can define a map $\delta : U(\mathfrak{sl}_2(\mathbb{C})) \to U(\mathfrak{sl}_2(\mathbb{C})) \otimes U(\mathfrak{sl}_2(\mathbb{C}))$ by

$$\delta(x) = \frac{\Delta_h(x_h) - \Delta_h^{\mathrm{op}}(x_h)}{h} \bmod h \tag{2.12.3}$$

where $x = x_h \bmod h$, for all $x_h \in U_h(\mathfrak{sl}_2(\mathbb{C}))$. In particular, we know that the classical Lie algebra elements, X, Y, and H are given by $X = X_h \bmod h$, $Y = Y_h \bmod h$ and $H = H_h$ respectively, so we may compute δ on these generators of $U(\mathfrak{sl}_2(\mathbb{C}))$ to obtain

$$\delta(X) = \frac{1}{2} X \wedge H, \tag{2.12.4}$$

$$\delta(Y) = \frac{1}{2} Y \wedge H, \tag{2.12.5}$$

$$\delta(H) = 0. \tag{2.12.6}$$

We see immediately that δ is a skew-symmetric map. Moreover we can check that for all $x \in \mathfrak{sl}_2(\mathbb{C})$, $\mathrm{CP}((\mathrm{id} \otimes \delta) \circ \delta(x)) = 0$ where $\mathrm{CP}(x_1 \otimes x_2 \otimes x_3) = x_1 \otimes x_2 \otimes x_3 + x_2 \otimes x_3 \otimes x_1 + x_3 \otimes x_1 \otimes x_2$. These conditions ensure that δ^* provides the vector space $\mathfrak{sl}_2(\mathbb{C})^*$ with a Lie algebra structure. Furthermore, it is a simple matter to check that

$$\delta([x,y]) = x \overset{\mathrm{ad}}{\triangleright} \delta(y) - y \overset{\mathrm{ad}}{\triangleright} \delta(x) \tag{2.12.7}$$

where the usual Lie algebra adjoint action, $\overset{\mathrm{ad}}{\triangleright}$, of $\mathfrak{sl}_2(\mathbb{C})$ on $\mathfrak{sl}_2(\mathbb{C})$ is extended to an action of $\mathfrak{g}$ on $\mathfrak{g} \otimes \mathfrak{g}$ in the usual way using the coproduct of $U(\mathfrak{sl}_2(\mathbb{C}))$. This says that δ is a 1-cocycle of $\mathfrak{sl}_2(\mathbb{C})$ with values in $\mathfrak{sl}_2(\mathbb{C}) \otimes \mathfrak{sl}_2(\mathbb{C})$, and, together with the two previous observations, tells us that $(\mathfrak{sl}_2(\mathbb{C}), \delta)$ is a Lie bialgebra according to the following general definition.

DEFINITION 2.12.3. A complex simple *Lie bialgebra* is a pair $(\mathfrak{g}, \delta)$ where $\mathfrak{g}$ is a complex simple Lie algebra and $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is a skew-symmetric linear map such that

1. δ is a 1-cocycle of $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$.
2. $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket on $\mathfrak{g}^*$.

Now δ was defined originally on the whole of $U(\mathfrak{sl}_2(\mathbb{C}))$, but we restricted attention to its action on the Lie algebra elements embedded in $U(\mathfrak{sl}_2(\mathbb{C}))$ to arrive at the Lie bialgebra $(\mathfrak{sl}_2(\mathbb{C}), \delta)$. In fact the pair $(U(\mathfrak{sl}_2(\mathbb{C})), \delta)$ is an example of a *co-Poisson Hopf algebra*.

DEFINITION 2.12.4. A *co-Poisson Hopf algebra* (A, δ) is a Hopf algebra A equipped with a skew-symmetric linear map $\delta : A \to A \otimes A$ such that

1. $\mathrm{CP} \circ (\delta \otimes \mathrm{id}) \circ \delta(a) = 0$ for all $a \in A$;
2. $(\Delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \Delta + (\mathrm{id} \otimes P) \circ (\delta \otimes \mathrm{id}) \circ \Delta$, where $P(a \otimes b) = b \otimes a$ for all $a, b \in A$;
3. $\delta(ab) = \delta(a)\Delta(b) + \Delta(a)\delta(b)$ for all $a, b \in A$.

As we would expect there is close relationship between co-Poisson Hopf algebras built on enveloping algebras $U(\mathfrak{g})$ and Lie bialgebras.

THEOREM 2.12.5. *Given a co-Poisson Hopf algebra* $(U(\mathfrak{g}), \delta)$ *then* $(\mathfrak{g}, \delta|_{\mathfrak{g}})$ *is a Lie bialgebra. Conversely, starting with any Lie bialgebra* $(\mathfrak{g}, \delta)$ *the cocommutator* δ *extends uniquely to provide* $U(\mathfrak{g})$ *with the structure of a co-Poisson Hopf algebra.*

The observation that $U_h(\mathfrak{sl}_2(\mathbb{C}))$ has a Lie bialgebra structure 'built in' at first order in h is not peculiar to $U_h(\mathfrak{sl}_2(\mathbb{C}))$.

THEOREM 2.12.6. *If* $U_h(\mathfrak{g})$ *is any QUEA — whether a standard Drinfeld-Jimbo one or not — then with* δ *defined as*

$$\delta(x) = \frac{\Delta_h(x_h) - \Delta_h^{\mathrm{op}}(x_h)}{h} \bmod h, \tag{2.12.8}$$

where $x = x_h \bmod h$, $U(\mathfrak{g}, \delta)$ *is a co-Poisson Hopf algebra and consequently* $(\mathfrak{g}, \delta|_{\mathfrak{g}})$ *is a Lie bialgebra.*

We can now formulate an alternative 'bottom up' definition of a QUEA.

DEFINITION 2.12.7. A *quantisation* of a Lie bialgebra $(\mathfrak{g}, \delta)$, as a quantised universal enveloping algebra, $U_h(\mathfrak{g})$, is a deformation of the Hopf algebra $U(\mathfrak{g})$ such that that the cocommutator δ of $(\mathfrak{g}, \delta)$ is given by the restriction to $\mathfrak{g}$ of that defined in (2.12.8).

2.13. Classical r-matrices

A first step towards classifying QUEAs of complex simple Lie algebras is clearly a classification of the complex simple Lie bialgebras. The cocommutator δ in each case is a 1-cocycle of $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$. But applying Whitehead's Lemma with $M = \mathfrak{g} \otimes \mathfrak{g}$ we see that every 1-cocycle must be a 1-coboundary. That is, the cocommutator must be of the form $\delta(x) = x \overset{\mathrm{ad}}{\triangleright} r$ for all $x \in \mathfrak{g}$ where r is some element of $\mathfrak{g} \otimes \mathfrak{g}$. In order that $(\mathfrak{g}, \delta)$ be a Lie bialgebra, it then follows that r must be such that

1. $\mathfrak{g} \overset{\mathrm{ad}}{\triangleright} (r_{12} + r_{21}) = 0$.
2. $\mathfrak{g} \overset{\mathrm{ad}}{\triangleright} ([[r, r]]) = 0$ where $[[r, r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]$.

Up to scalar multiplication there is only one $\mathfrak{g}$-invariant element of $\mathfrak{g} \otimes \mathfrak{g}$. This is the Casimir element C corresponding to the Killing form $\mathfrak{B}$ of $\mathfrak{g}$, so we deduce that

$$r_{12} + r_{21} = \kappa C, \tag{2.13.1}$$

where κ is some scalar. This means that we can write r in the form $r = \frac{1}{2}(r_{12} - r_{21}) + \frac{\kappa}{2}C$. But from $\delta(x) = x \overset{\mathrm{ad}}{\triangleright} r$ it follows that $r' = \frac{1}{2}(r_{12} - r_{21})$ defines the same bialgebra structure as $r = \frac{1}{2}(r_{12} - r_{21}) + \frac{\kappa}{2}C$ so we can concentrate on antisymmetric r-matrices.

Supposing r to be antisymmetric, it can be shown that $[[r, r]] \in \wedge^3\mathfrak{g}$ and furthermore that the $\mathfrak{g}$-invariant subspace of $\wedge^3\mathfrak{g}$ is 1-dimensional and spanned by $[[C, C]]$.

If r is skew and s is $\mathfrak{g}$-invariant then $[[r + s, r + s]] = [[r, r]] + [[s, s]]$. Thus we arrive at the following result through which we define the notion of a classical r-matrix.

THEOREM 2.13.1. *Every complex simple Lie bialgebra is coboundary and specified by an element* $r \in \mathfrak{g} \otimes \mathfrak{g}$ *which may be written as* $r = r_A + \kappa C$ *where* r_A *is an element of* $\mathfrak{g} \wedge \mathfrak{g}$ *and* κ *is a scalar such that* r *satisfies the classical Yang-Baxter equation* $[[r, r]] = 0$. *Such an element* r *is then called a* classical r-matrix.

We may now reflect that in Example 2.12.2 we ignored the quasitriangularity of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ when we considered its semi-classical limit. If we had taken it into account, then the definition of the co-commutator could have been written as

$$\delta(x) = \frac{\Delta_h(x_h) - \mathcal{R}\Delta_h(x_h)\mathcal{R}^{-1}}{h} \bmod h, \tag{2.13.2}$$

from which we would have obtained the results,

$$\delta(X) = X \overset{\mathrm{ad}}{\triangleright} r, \tag{2.13.3}$$

$$\delta(Y) = Y \overset{\mathrm{ad}}{\triangleright} r, \tag{2.13.4}$$

$$\delta(H) = H \overset{\mathrm{ad}}{\triangleright} r, \tag{2.13.5}$$

where $r \in \mathfrak{sl}_2(\mathbb{C}) \otimes \mathfrak{sl}_2(\mathbb{C})$ appears in the expansion $\mathcal{R} = 1 + hr \bmod h^2$.

By virtue of the fact that $\mathcal{R}$ obeys the quantum Yang-Baxter equation, r obeys the classical Yang-Baxter equation. Explicitly, it is given by

$$r = \frac{1}{4} H \otimes H + X \otimes Y, \tag{2.13.6}$$

and borrowing terminology from the quantum case we say that r defines $(\mathfrak{sl}_2(\mathbb{C}), \delta)$ as a *quasitriangular Lie bialgebra.* Indeed we have just seen that *every* complex simple Lie bialgebra is quasitriangular which in a sense 'explains' the quasitriangularity of the standard QUEAs.

A subset of topological quasitriangular Hopf algebras is the topological *triangular* Hopf algebras. These have universal R-matrices which satisfy $\mathcal{R}_{21}\mathcal{R} = \mathcal{I}$ where $\mathcal{I} = 1 \hat{\otimes} 1$. At the semi-classical level this is also a useful sub-division. The complex simple triangular Lie bialgebras are precisely those whose r-matrices are given by $r = r_A$.

2.14. Classification of classical r-matrices

We have already seen that there are standard quasitriangular QUEAs associated with all complex simple Lie algebras. The corresponding standard Lie bialgebras are described in terms of the so-called Drinfeld-Jimbo classical r-matrices, r^{DJ}. These are non-triangular solutions of the classical Yang-Baxter equation. If $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$, we denote by C_0 the restriction of the Casimir to $\mathfrak{h} \otimes \mathfrak{h}$. Then we can choose a basis element E_α for each root subspace $\mathfrak{g}_\alpha$ such that the Drinfeld-Jimbo r-matrices are given by

$$r^{DJ} = \frac{1}{2} C_0 + \sum_{\alpha \in \Delta^+} E_\alpha \otimes E_{-\alpha}. \tag{2.14.1}$$

These are by no means the only classical r-matrices for complex simple Lie algebras — there are many 'nonstandard' examples. The complete classification consists of two distinct classifications for the non-triangular and triangular cases respectively.

In the case of the non-triangular r-matrices the classification is due to Belavin and Drinfeld. To state their result we need a preliminary definition.

DEFINITION 2.14.1. A quadruple (Π_0, Π_1, τ, s) where Π_0 and Π_1 are subsets of the set Π of simple roots, $\tau : \Pi_0 \to \Pi_1$ a one-to-one mapping and $s \in \mathfrak{h} \wedge \mathfrak{h}$, is said to be *admissible* if it satisfies the following three conditions:

1. $(\tau(\alpha), \tau(\beta)) = (\alpha, \beta)$ for all $\alpha, \beta \in \Pi_0$.
2. For every $\alpha \in \Pi_0$, there is an n such that $\alpha, \tau(\alpha), \dots, \tau^{n-1}(\alpha) \in \Pi_0$ but $\tau^n(\alpha) \notin \Pi_0$.
3. $(\tau(\alpha) \otimes 1)(\frac{1}{2}C_0 + s) + (1 \otimes \alpha)(\frac{1}{2}C_0 + s) = 0$ for all $\alpha \in \Pi_0$.

Let us also introduce an ordering of the positive roots such that $\alpha < \beta$ if $\beta = \tau^m(\alpha)$ for some $m > 0$ (τ is extended linearly). The result of Belavin and Drinfeld may now be stated:

THEOREM 2.14.2. *If (Π_0, Π_1, τ, s) is an admissible quadruple then a Cartan-Weyl basis of the root subspaces of $\mathfrak{g}$ can be chosen such that*

$$r = r^{DJ} + s + \sum_{\alpha,\beta \in \Delta^+, \beta < \alpha} E_\alpha \wedge E_\beta \tag{2.14.2}$$

is a non-triangular classical r-matrix and moreover every non-triangular classical r-matrix is equivalent via a Lie algebra automorphism to one of this form.

For $\mathfrak{sl}_2(\mathbb{C})$ we see that the only non-triangular r-matrix is the Drinfeld-Jimbo solution. Somewhat more interesting is the situation for $\mathfrak{sl}_3(\mathbb{C})$. In this case there is also a non-standard non-triangular r-matrix. For reasons which should become clear in Chapter 3 we call it the Cremmer-Gervais r-matrix and denote it by r^{CG}. Explicitly

$$r^{CG} = r^{DJ} + \frac{1}{6} H_\alpha \wedge H_\beta + E_\alpha \wedge E_{-\beta}, \tag{2.14.3}$$

where α, β are the simple roots and the notation for the Lie algebra elements is that of Chapter 1.

It is worth setting this result beside the following observation. For any quasitriangular Lie bialgebra, $(\mathfrak{g}, r)$, and given any element $f \in \mathfrak{g} \otimes \mathfrak{g}$ satisfying

$$[[f_{21} - f_{12}, f_{21} - f_{12}]] + [[f_{21} - f_{12}, r]] + [[r, f_{21} - f_{12}]] = 0 \tag{2.14.4}$$

$(\mathfrak{g}, r + f_{21} - f_{12})$ is also a quasitriangular Lie bialgebra. We say that $(\mathfrak{g}, r + f_{21} - f_{12})$ is related to $(\mathfrak{g}, r)$ through a semiclassical twist by f. Notice that it follows that every quasitriangular Lie bialgebra is related via such a twisting to the trivial quasitriangular Lie bialgebra $(\mathfrak{g}, 0)$, so that in fact all the quasitriangular Lie bialgebras associated with a given Lie algebra are related amongst themselves through semiclassical twists. In particular, in the case of the Cremmer-Gervais r-matrix for $\mathfrak{sl}_3(\mathbb{C})$, we know that r^{DJ} and r^{CG} are related through a semiclassical twist $f = \frac{1}{6} H_\alpha \otimes H_\beta + E_\alpha \otimes E_{-\beta}$. Of course these observations are rather trivial. We mention them because the quantum counterparts of these twists are far from trivial, and indeed are far from being completely understood. One of the main results of Chapter 3 is actually a construction of a 'quantisation' of this twist.

The classification of the triangular solutions of the classical Yang-Baxter equation was achieved some time after the Belavin-Drinfeld result by Stolin. For details we refer the reader to the original papers [**94**]. Here we will just mention that for $\mathfrak{sl}_2(\mathbb{C})$ there is a single triangular r-matrix, r^J,

$$r^J = X \wedge H \tag{2.14.5}$$

which defines co-commutators

$$\delta(X) = 0, \tag{2.14.6}$$

$$\delta(Y) = 2X \wedge Y, \tag{2.14.7}$$

$$\delta(H) = 2X \wedge H. \tag{2.14.8}$$

There is a known quantisation of $(\mathfrak{sl}_2(\mathbb{C}), r^J)$ as a triangular QUEA, which is often called the Jordanian quantised universal enveloping algebra (hence the superscript). The Hopf algebra structure was discovered independently by Ohn [**81**] and Lazarev and Movshev [**66**] with the triangular universal R-matrix worked out subsequently by Ballesteros et al [**7**] and Abdesselam et al [**1**]. It will be described explicitly in Chapter 4, where we classify the bicovariant differential calculi on the corresponding quantised coordinate rings, $\mathbb{C}_h^J[GL_2(\mathbb{C})]$ and $\mathbb{C}_h^J[SL_2(\mathbb{C})]$.

Given the classification of the semiclassical structures it is now a question of whether or not there is a QUEA corresponding to each complex simple Lie bialgebra which contains the Lie bialgebra in its semi-classical limit. There is an early result of Drinfeld which establishes the existence and uniqueness up to 'quantum' twists of a triangular QUEA quantising every triangular Lie bialgebra. The existence problem in the non-triangular case has only been resolved quite recently by Etingof and Kahzdan [**38**]. The correct mathematical framework for discussing such issues is provided by Drinfeld's notion of quasi-Hopf algebras. Uniqueness does not seem to have been established for any but the standard Drinfeld-Jimbo quantum groups.

2.15. Quasi-Hopf algebras and twisting

Given a topological quasitriangular Hopf algebra $(U_h(\mathfrak{g}), \mathcal{R})$ and a $U_h(\mathfrak{g})$-module, $V[[h]]$, the form of the universal R-matrix ensures that $\mathcal{R}$ is a well defined operator on $(V \otimes V)[[h]]$. In fact this is clear in the case of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ with $\mathcal{R}$ as defined in (2.5.16) since by Theorem 2.6.14 the elements X and Y act nilpotently on $V[[h]]$. The same reasoning applies in the general case.

EXAMPLE 2.15.1. Choose V to be the first fundamental representation of $U(\mathfrak{sl}_2(\mathbb{C}))$ and take $\{v_0, v_1\}$ as a basis for $V[[h]]$ as in Theorem 2.6.14 so that

$$\{v_0 \otimes v_0, v_0 \otimes v_1, v_1 \otimes v_0, v_1 \otimes v_1\} \tag{2.15.1}$$

is a basis of $(V \otimes V)[[h]]$. On $(V \otimes V)[[h]]$ the universal R-matrix of $U_h(\mathfrak{sl}_2(\mathbb{C}))$, $\mathcal{R}$, then acts as the matrix $R_{V,V}$ where

$$R_{V,V} = e^{-h/4} \begin{pmatrix} e^{h/2} & 0 & 0 & 0 \\ 0 & 1 & e^{h/2} - e^{-h/2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{h/2} \end{pmatrix}. \tag{2.15.2}$$

In this example $R_{V,V}$ is a $\mathbb{C}[[h]]$-linear automorphism of $(V \otimes V)[[h]]$. This is the case generally when we denote by $R_{V,V}$ the universal R-matrix interpreted as an operator on $(V \otimes V)[[h]]$ where $V[[h]]$ is a $U_h(\mathfrak{g})$-module since $\mathcal{R}$ is by definition invertible. Moreover, defining $\hat{R}_{V,V} = \tau_{V,V} \circ R_{V,V}$ where $\tau_{V,V} : (V \otimes V)[[h]]$ is the $\mathbb{C}[[h]]$-linear permutation map such that $\tau_{V,V}(v \hat{\otimes} w) = w \hat{\otimes} v$ for all $v, w \in V[[h]]$, the QYBE implies that $\hat{R}_{V,V}$ satisfies the *braid relation*

$$(\hat{R}_{V,V} \hat{\otimes} \mathrm{id}) \circ (\mathrm{id} \hat{\otimes} \hat{R}_{V,V}) \circ (\hat{R}_{V,V} \hat{\otimes} \mathrm{id}) = (\mathrm{id} \hat{\otimes} \hat{R}_{V,V}) \circ (\hat{R}_{V,V} \hat{\otimes} \mathrm{id}) \circ (\mathrm{id} \hat{\otimes} \hat{R}_{V,V}). \quad (2.15.3)$$

Let us now recall the definition of the braid group.

DEFINITION 2.15.2. The *braid group on n strands B_n* is the group generated by the $n-1$ symbols $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ such that $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| > 1$ and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad (2.15.4)$$

for all $i, j = 1 \ldots n-1$.

Given the formal similarity between (2.15.3) and (2.15.4) it is not surprising that there is a simple construction which turns the operators $\hat{R}_{V,V}$ into $\mathbb{C}[[h]]$-linear representations of the braid group B_n. Indeed defining operators b_i on $V^{\otimes n}[[h]]$ for $i = 1 \ldots n-1$ by

$$b_i = \begin{cases} \hat{R}_{V,V} \hat{\otimes} \mathrm{id}_{V^{\otimes (n-2)}[[h]]} & i = 1, \\ \mathrm{id}_{V^{\otimes (i-1)}[[h]]} \hat{\otimes} \hat{R}_{V,V} \hat{\otimes} \mathrm{id}_{V^{\otimes (n-i-1}[[h]]} & 1 < i < n-1, \\ \mathrm{id}_{V^{\otimes (n-2)}[[h]]} \hat{\otimes} \hat{R}_{V,V}, & \end{cases} \quad (2.15.5)$$

then we have precisely $b_i b_j = b_j b_i$ if $|i-j| > 1$ and

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, \quad (2.15.6)$$

for all $i, j = 1 \ldots n-1$.

Apparently many 'mathematical-physics-miles' from the discussion to this point are Wess-Zumino-Witten (WZW) theories in conformal field theory and the Knizhnik-Zamolodchikov (KZ) equations. In fact there is a KZ system of equations associated with any pair $(\mathfrak{g}, t)$ where $\mathfrak{g}$ is a complex simple Lie algebra and t is a symmetric $\mathfrak{g}$-invariant element of $\mathfrak{g} \otimes \mathfrak{g}$ together with a parameter h an integer $n \geq 1$ and a $\mathfrak{g}$-module V. Moreover, through a classical construction of the monodromy of the the KZ equations is obtained a representation of the braid group B_n called the monodromy representation which, for $n = 2$, has the form $\tau_{V,V} \circ e^{ht/2}$.

Kohno and Drinfeld independently obtained the following remarkable result:

THEOREM 2.15.3. *Let $\mathfrak{g}$ be a complex simple Lie algebra and V an irreducible $\mathfrak{g}$-module. Then the monodromy representation of the braid group B_n on $V^{\otimes n}[[h]]$ from the KZ equations is equivalent to the $\mathbb{C}[[h]]$-linear representations of the braid group determined by the universal $\mathcal{R}$ of the Drinfeld-Jimbo QUEA $U_h(\mathfrak{g})$.*

In a series of fundamental works [**33, 35, 36**] Drinfeld simultaneously proved this theorem, introduced new mathematical structures generalising his earlier innovation of quasitriangular Hopf algebras, and set up the correct framework for questions of existence and uniqueness of quantum groups. His new mathematical structures will now be defined.

DEFINITION 2.15.4. A *topological quasi-Hopf algebra over $\mathbb{C}[[h]]$* is a a topological algebra, (A, m, η), together with $\mathbb{C}[[h]]$-linear maps $\Delta : A \to A \hat{\otimes} A$, $\epsilon : A \to \mathbb{C}[[h]]$ and

$S : A \to A$ called the coproduct, counit and antipode respectively and an invertible element $\Phi \in A\hat{\otimes}A\hat{\otimes}A$ called the *Drinfeld associator* such that

$$(\mathrm{id}\,\hat{\otimes}\Delta) \circ \Delta(a) = \Phi^{-1}((\Delta\hat{\otimes}\,\mathrm{id}) \circ \Delta(a))\Phi, \tag{2.15.7}$$

$$((\Delta\hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\,\mathrm{id})(\Phi))((\mathrm{id}\,\hat{\otimes}\,\mathrm{id}\,\hat{\otimes}\Delta)(\Phi)) = (\Phi\hat{\otimes}1)((\mathrm{id}\,\hat{\otimes}\Delta\hat{\otimes}\,\mathrm{id})(\Phi))(1\hat{\otimes}\Phi), \tag{2.15.8}$$

$$(\epsilon\hat{\otimes}\,\mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id}\,\hat{\otimes}\epsilon) \circ \Delta, \tag{2.15.9}$$

$$(\mathrm{id}\,\epsilon\hat{\otimes}\,\mathrm{id})(\Phi) = 1, \tag{2.15.10}$$

$$S(a_{(1)})\alpha a_{(2)} = \epsilon(a)\alpha, \quad a_{(1)}\beta S(a_{(2)}) = \epsilon(a)\beta, \tag{2.15.11}$$

$$\sum S(\phi)\alpha\phi'\beta S(\phi'') = 1, \quad \sum \overline{\phi}\beta S(\overline{\phi}')\alpha\overline{\phi}'' = 1, \tag{2.15.12}$$

for all $a \in A$ where $\Delta(a) = a_{(1)}\hat{\otimes}a_{(2)}$ is a Sweedler notation for the coproduct, 1 is the unit in A, $\Phi = \sum \phi\hat{\otimes}\phi'\hat{\otimes}\phi''$ and $\Phi^{-1} = \sum \overline{\phi}\hat{\otimes}\overline{\phi}'\hat{\otimes}\overline{\phi}''$.

DEFINITION 2.15.5. A *topological quasitriangular quasi-Hopf algebra* (QTQHA) is a tuple $(A, m, \eta, \Delta, \epsilon, S, \alpha, \beta, \Phi, \mathcal{R})$ where $(A, m, \eta, \Delta, \epsilon, S, \alpha, \beta, \Phi)$ is a quasi-Hopf algebra and $\mathcal{R}$ is invertible and called the universal R-matrix such that

$$\Delta^{\mathrm{op}}(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}, \tag{2.15.13}$$

$$(\Delta\hat{\otimes}\,\mathrm{id})(\mathcal{R}) = \Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{132}\mathcal{R}_{23}\Phi_{123}^{-1}, \tag{2.15.14}$$

$$(\mathrm{id}\,\hat{\otimes}\Delta)(\mathcal{R}) = \Phi_{312}\mathcal{R}_{13}\Phi_{213}^{-1}\mathcal{R}_{12}\Phi_{123} \tag{2.15.15}$$

for all $a \in A$.

These definitions are imposing to say the least and do not suggest that any conceptual simplification will be obtained by working with QTQHAs rather than quasitriangular Hopf algebras. However the QTQHAs have a truly fundamental property which is not in general shared by the quasitriangular Hopf algebras. They admit *twists* according to the following theorem:

THEOREM 2.15.6. *Let $(A, m, \eta, \Delta, \epsilon, S, \alpha, \beta, \Phi, \mathcal{R})$ be a QTQHA and suppose $\mathcal{F} \in A\hat{\otimes}A$ is invertible and such that*

$$(\epsilon\hat{\otimes}\,\mathrm{id})(\mathcal{F}) = 1 = (\mathrm{id}\,\hat{\otimes}\epsilon)(\mathcal{F}); \tag{2.15.16}$$

then $\mathcal{F}$ is called a universal twisting matrix *and $(A_{\mathcal{F}}, m, \eta, \Delta_{\mathcal{F}}, \epsilon, S, \alpha, \beta, \Phi_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}})$ where,*

$$\Delta_{\mathcal{F}}(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1}, \tag{2.15.17}$$

$$\mathcal{R}_{\mathcal{F}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}_{21}^{-1}, \tag{2.15.18}$$

$$\Phi_{\mathcal{F}} = \mathcal{F}_{12}((\Delta\hat{\otimes}\,\mathrm{id})(\mathcal{F}))\Phi((\mathrm{id}\,\hat{\otimes}\Delta)(\mathcal{F}))^{-1}\mathcal{F}_{23}^{-1} \tag{2.15.19}$$

is a QTQHA.

Observe that any topological (quasitriangular) Hopf algebra is a (QT)QHA for which the Drinfeld associator is just $1\hat{\otimes}1\hat{\otimes}1$. Also, it is of course the case that the definitions and twisting result apply equally well if the objects are defined over a field such as $\mathbb{C}$ rather than the ring $\mathbb{C}[[h]]$.

The twisting result may be specialised to the interesting case of twisting from and to a quasitriangular Hopf algebra. Such twists (or rather their duals) will be the subject of Chapter 3.

DEFINITION 2.15.7. A *quasi-Hopf quantised universal enveloping algebra* (QHQUEA) associated to a complex simple Lie algebra $\mathfrak{g}$ is a QHA $(A, m, \eta, \Delta, \epsilon, S, \alpha, \beta, \Phi)$ such that $A/hA \cong U(\mathfrak{g})$ as quasi-Hopf algebras and $A \cong U(\mathfrak{g})[[h]]$ as $\mathbb{C}[[h]]$-modules and $\Phi = 1 \otimes 1 \otimes 1 \bmod h^2$. A quasitriangular QHQUEA is a QHQUEA which is also a topological quasitriangular quasi Hopf algebra with $\mathcal{R} = 1 \otimes 1 \bmod h$.

DEFINITION 2.15.8. The *classical limit* of a quasitriangular QHQUEA $(A, \Phi, \mathcal{R})$ is the pair $(\mathfrak{g}, t)$ where $t = h^{-1}(\mathcal{R}_{21}\mathcal{R}_{12} - 1 \otimes 1) \bmod h$. A quasitriangular QHQUEA with $(\mathfrak{g}, t)$ as its classical limit is then called a *quantisation* of $(\mathfrak{g}, t)$.

Drinfeld's fundamental result is the following:

THEOREM 2.15.9. *Given any Lie algebra, $\mathfrak{g}$, together with a symmetric $\mathfrak{g}$-invariant element, t, there exists a quantisation of the universal enveloping algebra, $U(\mathfrak{g})$, as a* quasitriangular quasi-Hopf quantised universal enveloping algebra, *$(U(\mathfrak{g})[[h]], \Phi_{KZ}, e^{ht/2})$, and this quantisation is* unique *amongst quantisations of $(\mathfrak{g}, t)$ up to twisting.*

An immediate consequence of this result is that the standard quantisation of $U(\mathfrak{sl}_{l+1}(\mathbb{C}))$, $(U_h(\mathfrak{sl}_{l+1}(\mathbb{C})), \mathcal{R}_S)$, is twist equivalent as a *quasitriangular quasi-Hopf algebra* to the 'universal' quantisation $(U(\mathfrak{sl}_{l+1}(\mathbb{C}))[[h]], \Phi, e^{ht/2})$. We consider the relevance of this result in relation to the Belavin-Drinfeld classification of classical r-matrices and recent work of Etingof and Kazhdan in Chapter 3.

CHAPTER 3

Twisting 2-cocycles for the construction of new non-standard quantum groups

3.1. Introduction

Originally there were two clearly defined types of quantum groups [**32, 55, 84**]. They were *single-parameter* quantisations, $U_h(\mathfrak{g})$ and $\mathbb{C}_h[G]$ respectively, of dual classical objects: the universal enveloping algebras of simple Lie algebras, $U(\mathfrak{g})$, and the coordinate rings of simple Lie groups, $\mathbb{C}[G]$. With their universal R-matrices, $\mathcal{R}$, these $U_h(\mathfrak{g})$ are the standard examples of quasitriangular Hopf algebras, while the $\mathbb{C}_h[G]$, together with the corresponding numerical R-matrices, are the standard examples of what we call co-quasitriangular Hopf algebras [**29, 30**]. It soon became apparent that there were a number of multiparameter generalisations [**26, 95, 99**] of these standard quantum groups and through the work of Drinfeld [**33**], followed by Reshetikhin [**83**], an interpretation emerged: all the multiparameter quantum groups corresponding to a particular standard quantum group were related, amongst themselves and with the standard quantum group, through Drinfeld's important process of twisting. In fact the original works of Drinfeld and Reshetikhin were concerned only with quasitriangular Hopf algebras, but their constructions dualise immediately to the case of co-quasitriangular Hopf algebras. Since the twists act only as similarity transformations on the so called $\hat{R}$-matrices [**84**], the different standard quantum groups corresponding to different classical Lie groups cannot be related to each other by twisting. The picture then is of a number of distinct 'twist equivalence classes'. Later, Kempf and Engeldinger [**61, 37**] (see also the work of Khoroshkin and Tolstoy [**62**]) refined Reshetikhin's work slightly and showed that there were other interesting quantum groups related, through Reshetikhin-type twists, with the standard ones.

From time to time there appeared genuinely non-standard quantum groups, usually defined in terms of non-standard numerical R-matrices. It is natural to investigate whether these define new twist equivalence classes or whether they belong to classes already defined by the standard R-matrices. We will be particularly concerned in this article with the non-standard quantum groups of Cremmer and Gervais [**22**] and Fronsdal and Galindo [**41, 42**] which for general theoretical reasons (see Section 4) may be expected to be twist-equivalent to the standard $SL(n)$ quantum groups. However, let us be clear that *no* twist of the Reshetikhin type is suitable in these cases. As we explain later, the relevant twisting structures are counital 2-cocycles on the quantum groups. The problem then is to find the appropriate twisting 2-cocycles, defined on the standard co-quasitriangular Hopf algebra, $\mathbb{C}_h[SL(n)]$, which twist this quantum group into the Cremmer-Gervais and Fronsdal-Galindo quantum groups.

Recently, Hodges has made significant progress in this area [**50**], drawing on previous work of his contained in a series of important papers [**51, 52, 53**]. These covered many aspects of quantum group theory from a ring theoretic perspective. We should mention [**52**] in particular, where some remarkable aspects of the algebraic structure of Cremmer-Gervais quantum groups were revealed. In [**50**] Hodges starts from a particular, standard, multiparameter quantised enveloping algebra $U_p(\mathfrak{g})$ (this is a multiparameter version of a rational form $U_q(\mathfrak{g})$ of $U_h(\mathfrak{g})$). He then identifies a pair of *commuting* sub-Hopf algebras, $U_p(\mathfrak{b}_1^-)$ and $U_p(\mathfrak{b}_2^+)$, associated with certain Belavin-Drinfeld triples [**10**]. This gives rise to a Hopf algebra homomorphism, $\phi : U_p(\mathfrak{b}_1^-) \otimes U_p(\mathfrak{b}_2^+) \to U_p(\mathfrak{g})$, through the usual multiplication map. Attention then shifts to the dual map, $\phi^* : \mathbb{C}_p[G] \to \mathbb{C}_p[B_1^-] \otimes \mathbb{C}_p[B_2^+]$. Hodges proceeds to identify $\mathrm{Im}(\phi^*)$, in a series of precise and subtle steps, with the image of the tensor product of a pair of 'extended' Borel subalgebra-like objects, between which there is a skew pairing. This skew pairing lifts to the tensor components of $\mathrm{Im}(\phi^*)$. It is well known that such a pairing gives rise to a 2-cocycle, the quintessential example appearing in the twisting interpretation of the quantum double [**30, 71**], and a 2-cocycle is then induced on the quantised function algebra $\mathbb{C}_p[G]$. Hodges claims that in the particular case of $\mathfrak{sl}_3(\mathbb{C})$ the 2-cocycle coming from his construction generates the Cremmer-Gervais deformation of $\mathbb{C}[SL(3)]$. More generally, he claims that it should also be possible to reach the esoteric quantum groups of Fronsdal and Galindo [**41, 42**]. However, Hodges' approach is rather technical and does not readily yield 2-cocycles defined *explicitly* on the familiar T-matrix generators of standard quantum groups. We are able to remedy this situation here.

Our approach is actually quite distinct from that of Hodges. We work entirely within the framework of co-quasitriangular Hopf algebras coming from solutions of the matrix quantum Yang-Baxter equation (QYBE). Since we are working with quantum groups coming from the FRT construction and are not concerned with their relationship with QUEAs we do not need to work in a topological setting. Indeed from now on we denote the quantised coordinate rings coming from the FRT construction by $\mathbb{C}_q[G]$ where q is an element of some base field k and in particular (standard) cases can be taken to be $e^{h/2}$.

In Section 2 we recall the definition of a co-quasitriangular Hopf algebra and the basic result on twisting by 2-cocycles. A good reference for this theory, and much more besides, is the book by Majid [**71**], from which much of our notation is borrowed. Other good references are the paper by Larson and Towber [**65**] and the papers of Doi and Takeuchi [**29, 30**]. Majid calls co-quasitriangular Hopf algebras, 'dual quasitriangular Hopf algebras', but our terminology comes from [**29, 30**]. We go on to describe the well known class of 2-cocycles which appears as the dual of the Reshetikhin-type twists. They originate from particular solutions of the QYBE. The 'parameterization' twists originally considered by Reshetikhin [**83**] may be regarded as examples in this class, and using such a twist we have obtained a *new* 3-parameter generalised Cremmer-Gervais R-matrix, presented here, which includes as a special case the 2-parameter R-matrix considered by Hodges in [**52**]. Details of the derivation of this new R-matrix are given in Appendix A. The sub-Hopf-algebra-induced twists considered by Engeldinger and Kempf [**61, 37**] also belong to this general class of 2-cocycles, and are recalled here.

Section 3 contains our main new results. We present there a new class of 2-cocycles which no longer emanate from solutions of the matrix QYBE. Instead, they arise from matrices satisfying a new, and remarkably simple, system of equations. A number of explicit 2-cocycles belonging to our new class are presented, along with the following results:

- It is shown *explicitly* that the *new* 3-parameter generalised Cremmer-Gervais quantum group corresponding to $GL(3)$, already given in Section 2, is obtained from a particular multiparameter standard quantum group through twisting.
- The 2-cocycle used to obtain the generalised Cremmer-Gervais deformation of $GL(3)$ is an example of a general class of *simple root* 2-cocycles which themselves belong to a more general class of *composite simple root* 2-cocycles. These 2-cocycles may all be defined on certain standard, multiparameter, deformations of $GL(n)$, and consequently generate new non-standard quantum groups.
- A 2-cocycle which can be used to twist a certain multiparameter standard deformation of $\mathbb{C}[GL(2N-1)]$ to obtain a *generalisation* of the quantum groups of Fronsdal and Galindo is presented. For $N = 2$ this 2-cocycle is just the one used to obtain the generalised Cremmer-Gervais $GL(3)$ quantum group.

Note that the R-matrix considered by Fronsdal and Galindo is already 'multiparameter', involving N parameters, but we obtain, in Appendix B, an R-matrix which depends on $(1 + \frac{1}{2}(N-1)(N+2))$ parameters. Let us also note that, as was already suggested in Hodges work [**50**], starting from the original standard quantum groups, $\mathbb{C}_q[SL(n)]$, we need a *combination* of the Reshetikhin-type parameterisation twists with our new twists to obtain the Fronsdal-Galindo quantum groups.

In Section 4, we collect some information about the semi-classical objects corresponding to the R-matrices which we have been considering in this chapter, namely the classical r-matrices. We also recall the background, in Drinfeld's fundamental work, which serves as the on-going motivation in the quest for interesting twists. We end by pointing out a particular problem involved in constructing the Cremmer-Gervais quantum group corresponding to $GL(4)$, and describe an interesting phenomenon involving sub-Hopf-algebra-induced twists. Starting from the standard multiparameter quantum group, $\mathbb{C}_{q,p}[GL(4)]$, we twist, first of all, using a sub-Hopf-algebra-induced twist. The resulting quantum group may reasonably be called 'weakly non-standard' and can be twisted further using a 2-cocycle from our construction. The new, non-standard, R-matrix obtained through this double twist involves a pair of non-standard off-diagonal elements which could not have been added to the original R-matrix directly, but which *do* appear in the Cremmer-Gervais R-matrix for $GL(4)$.

3.2. Co-quasitriangular Hopf algebras and Reshetikhin twists

We begin with some basic definitions.

DEFINITION 3.2.1. A bialgebra A is called *co-quasitriangular* if there exists a bilinear form σ on A, which we will call an R-form, such that

1. σ is invertible with respect to the convolution product, $*$, that is, there is another bilinear form σ^{-1} such that
$$\sigma(a_{(1)}, b_{(1)})\sigma^{-1}(a_{(2)}, b_{(2)}) = \epsilon(ab) = \sigma^{-1}(a_{(1)}, b_{(1)})\sigma(a_{(2)}, b_{(2)}), \tag{3.2.1}$$
2. $\sigma * m = m^{\mathrm{op}} * \sigma$, i.e.
$$\sigma(a_{(1)}, b_{(1)})a_{(2)}b_{(2)} = b_{(1)}a_{(1)}\sigma(a_{(2)}, b_{(2)}), \tag{3.2.2}$$

3. $\sigma(m \otimes \mathrm{id}) = \sigma_{13} * \sigma_{23}$, i.e.
$$\sigma(ab, c) = \sigma(a, c_{(1)})\sigma(b, c_{(2)}), \tag{3.2.3}$$
4. $\sigma(\mathrm{id} \otimes m) = \sigma_{13} * \sigma_{12}$, i.e.
$$\sigma(a, bc) = \sigma(a_{(1)}, c)\sigma(a_{(2)}, b). \tag{3.2.4}$$

REMARK 3.2.2. We are employing here a slightly simplified version of the Sweedler notation for coproducts: $\Delta(a) = a_{(1)} \otimes a_{(2)}$, with the summation suppressed.

REMARK 3.2.3. It may be useful to briefly recall how we arrive at this definition. Suppose that $(H, \mathcal{R})$ is a quasitriangular bialgebra, with $\mathcal{R} \in H \otimes H$ the *universal R-matrix* obeying Drinfeld's familiar axioms
$$(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad (\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}, \tag{3.2.5}$$
$$\Delta^{\mathrm{op}}(h) = \mathcal{R} \circ \Delta(h) \circ \mathcal{R}^{-1}, \quad \forall h \in H. \tag{3.2.6}$$
In fact, we may be regard $\mathcal{R}$ as a map $k \to H \otimes H$, where k is the ground field. When formulating the dual notion of co-quasitriangular bialgebra, we then need to consider an R-form σ which is now a map $A \otimes A \to k$, where A can be thought of as dual to H. To formulate the dual axioms, involving σ instead of $\mathcal{R}$, we require the algebra structure on $\mathrm{Hom}(A \otimes A, k)$. This is the convolution algebra provided by the natural tensor product coalgebra structure of $A \otimes A$. Explicitly then, let us give the details for a particular example,
$$\begin{aligned}(\sigma_{13} * \sigma_{23})(a \otimes b \otimes c) &= \sigma_{13}(a_{(1)} \otimes b_{(1)} \otimes c_{(1)})\sigma_{23}(a_{(2)} \otimes b_{(2)} \otimes c_{(2)}) \\ &= \sigma(a_{(1)}, c_{(1)})\epsilon(b_{(1)})\sigma(b_{(2)}, c_{(2)})\epsilon(a_{(2)}) \\ &= \sigma(a, c_{(1)})\sigma(b, c_{(2)}).\end{aligned}$$
For more on this process of 'dualising' we refer the reader to Majid's book [**71**], and his paper [**69**].

From the definition it is readily seen that the QYBE now manifests itself as
$$\sigma_{12} * \sigma_{13} * \sigma_{23} = \sigma_{23} * \sigma_{13} * \sigma_{12}. \tag{3.2.7}$$
When A is actually a Hopf algebra, with antipode S, we call it a *co-quasitriangular Hopf algebra.* It can then be shown that S is *always* invertible, and $\sigma^{-1}(a, b) = \sigma(S(a), b)$ and $\sigma(a, b) = \sigma^{-1}(a, S(b))$.

The FRT bialgebras $A(R)$, introduced by the Leningrad school [**84**] and developed by Majid [**70**], where R is any matrix solution of the QYBE, fit into the co-quasitriangular bialgebra framework. Indeed, we define the R-form on the generators T_i^j, as
$$\sigma(T_i^s, T_j^t) = R_{ij}^{st}, \tag{3.2.8}$$
or in the useful 'matrix notation', as
$$\sigma(T_1, T_2) = R_{12}, \tag{3.2.9}$$
and then extend its domain of definition to the whole of $A(R)$ by setting
$$\sigma(T_1T_2, T_3) = \sigma(T_1, T_3)\sigma(T_2, T_3) = R_{13}R_{23}, \tag{3.2.10}$$
$$\sigma(T_1, T_2T_3) = \sigma(T_1, T_3)\sigma(T_1, T_2) = R_{13}R_{12}. \tag{3.2.11}$$
The QYBE then guarantees consistency with the product relation (3.2.2).

REMARK 3.2.4. To remove any possible doubt about the notation being employed here, let us present (3.2.10) explicitly, in terms of the generators, as

$$\sigma(T_i^s T_j^t, T_k^r) = \sigma(T_i^s, T_k^m)\sigma(T_j^t, T_m^r) = R_{ik}^{sm} R_{jm}^{tr},$$

where the summation convention *is* being assumed.

DEFINITION 3.2.5. A bilinear form χ on a bialgebra A is called a *counital 2-cocycle* on A if it is invertible in the convolution product, and

$$\chi(1, a) = \epsilon(a) = \chi(a, 1), \tag{3.2.12}$$

and

$$\chi_{12} * \chi(m \otimes \mathrm{id}) = \chi_{23} * \chi(\mathrm{id} \otimes m). \tag{3.2.13}$$

REMARK 3.2.6. It is a simple matter to show that any R-form is a counital 2-cocycle. We also note that for any Hopf algebra on which we can define such a 2-cocycle, which moreover intertwines the multiplication as in (3.2.2), the antipode is necessarily invertible.

REMARK 3.2.7. In the more familiar dual version of this definition, we consider an invertible element $\mathcal{F}$ of $H \otimes H$. Then (3.2.12) and (3.2.13) correspond respectively to

$$(\epsilon \otimes \mathrm{id})(\mathcal{F}) = 1 = (\mathrm{id} \otimes \epsilon)(\mathcal{F}), \tag{3.2.14}$$

and

$$\mathcal{F}_{12}(\Delta \otimes \mathrm{id})(\mathcal{F}) = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta)(\mathcal{F}). \tag{3.2.15}$$

An element $\mathcal{F}$ satisfying these conditions is then called a counital 2-cocycle *for* H.

REMARK 3.2.8. For a general discussion of cocycles for and on Hopf algebras we refer the reader to Section 2.3 of Majid's book [**71**].

The property of co-quasitriangular Hopf algebras which is of particular interest to us is that, given one, we may generate others using these counital 2-cocycles. This important process of *twisting* is the dual of Drinfeld's original quasitriangular quasi-Hopf algebra twist [**33**], restricted to the special case of twisting *from and to* co-quasitriangular Hopf algebras. It is not difficult to dualise Drinfeld's original quasitriangular quasi-Hopf algebra axioms, and his result on twisting. This was probably first carried out explicitly by Majid [**69**]. We obtain the axioms for a co-quasitriangular co-quasi-Hopf algebra and on specialising the twisting result, we obtain the following important theorem.

THEOREM 3.2.9. *Let $(A, m, \eta, \Delta, \epsilon, \sigma)$ be a co-quasitriangular bialgebra and let χ be a counital 2-cocycle on A, then there is a new co-quasitriangular bialgebra (A_χ, σ_χ) obtained by twisting the product and R-form of (A, σ) as*

$$m_\chi = \chi * m * \chi^{-1}, \tag{3.2.16}$$
$$\sigma_\chi = \chi_{21} * \sigma * \chi^{-1}. \tag{3.2.17}$$

If A is moreover a Hopf algebra with antipode S, then A_χ is also a Hopf algebra with twisted antipode given by

$$S_\chi = \lambda * S * \lambda^{-1}, \tag{3.2.18}$$

where $\lambda = \chi \circ (\mathrm{id} \otimes S) \circ \Delta$.

For the co-quasitriangular bialgebras, $A(R)$, there is a particularly obvious way of constructing twisting 2-cocycles. Take any invertible solution F of the QYBE and define a bilinear form χ by

$$\chi(T_1, T_2) = F_{12}, \tag{3.2.19}$$

$$\chi(1, T) = \chi(T, 1) = \epsilon(T), \tag{3.2.20}$$

and

$$\chi^{-1}(T_1, T_2) = F_{12}^{-1}. \tag{3.2.21}$$

We then extend this to the whole of $A(R)$ just as we did for the R-form in equations (3.2.10) and (3.2.11), that is

$$\chi(T_1T_2, T_3) = \chi(T_1, T_3)\chi(T_2, T_3) = F_{13}F_{23}, \tag{3.2.22}$$

$$\chi(T_1, T_2T_3) = \chi(T_1, T_3)\chi(T_1, T_2) = F_{13}F_{12}. \tag{3.2.23}$$

However χ must respect the algebra structure already on $A(R)$ so we must also have

$$\chi(R_{12}T_1T_2 - T_2T_1R_{12}, T_3) = 0$$

$$\iff R_{12}F_{13}F_{23} = F_{23}F_{13}R_{12}, \tag{3.2.24}$$

and

$$\chi(T_1, R_{23}T_2T_3 - T_3T_2R_{23}) = 0$$

$$\iff R_{23}F_{13}F_{12} = F_{12}F_{13}R_{23}. \tag{3.2.25}$$

Thus any invertible solution F of the QYBE which satisfies (3.2.24) and (3.2.25) provides a 2-cocycle twist. Such a twisting system (R, F) may reasonably be called a *Reshetikhin twist* [**83**].

REMARK 3.2.10. In the context of the papers [**83, 61, 37**], where the approach is dual to ours, an invertible element $\mathcal{F} \in H \otimes H$ is considered, where $(H, \mathcal{R})$ is a quasitriangular Hopf algebra. It is assumed to satisfy the QYBE

$$\mathcal{F}_{12}\mathcal{F}_{13}\mathcal{F}_{23} = \mathcal{F}_{23}\mathcal{F}_{13}\mathcal{F}_{12}, \tag{3.2.26}$$

and the relations

$$(\Delta \otimes \mathrm{id})(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{23}, \tag{3.2.27}$$

and

$$(\mathrm{id} \otimes \Delta)(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{12}, \tag{3.2.28}$$

which correspond respectively to equations (3.2.22) and (3.2.23). This $\mathcal{F}$ is then a 2-cocycle for H in the sense of Remark 2.7, and twists the *comultiplication*, *universal R-matrix* and antipode as

$$\Delta_{\mathcal{F}}(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}. \quad \forall h \in H. \tag{3.2.29}$$

$$\mathcal{R}_{\mathcal{F}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}, \tag{3.2.30}$$

and

$$S_{\mathcal{F}}(h) = vS(h)v^{-1}, \quad \forall h \in H, \tag{3.2.31}$$

where $v = m \circ (\mathrm{id} \otimes S)(\mathcal{F})$. This is actually a slight generalisation of the presentation of Reshetikhin [**83**], due to Kempf and Engeldinger [**61, 37**].

We will be particularly interested in the situation pertaining when we take R to be the standard $SL(n)$ type R-matrix given by

$$(R_S)^{st}_{ij} = \begin{cases} q & i = j = s = t, \\ 1 & i = s \neq j = t, \\ (q - q^{-1}) & i = t < j = s. \end{cases} \tag{3.2.32}$$

In this case, with the identification of the central quantum determinant, $A(R)$ becomes a Hopf algebra. Indeed it is the standard quantisation of the coordinate ring of the Lie group $SL(n)$, denoted $\mathbb{C}_q[SL(n)]$.

Let us also present here the expressions, in our notation, for the Cremmer-Gervais $SL(n)$ R-matrix and the Fronsdal-Galindo $GL(2N-1)$ R-matrix. First, the Cremmer-Gervais R-matrix will be taken to be

$$(R_{CG})^{st}_{ij} = \begin{cases} q & i = j = s = t, \\ qq^{-2(j-s)/n} & i = s < j = t, \\ q^{-1}q^{-2(j-s)/n} & i = s > j = t, \\ (q - q^{-1}) & i = t < j = s, \\ (q - q^{-1})q^{-2(j-s)/n} & i < s < j, \text{ and } t = i + j - s, \\ -(q - q^{-1})q^{-2(j-s)/n} & j < s < i, \text{ and } t = i + j - s. \end{cases} \tag{3.2.33}$$

If the R-matrix which appears as equation (46) in the original paper of Cremmer and Gervais [**22**] is denoted $\tilde{R}$ then $R_{CG} = \tilde{R}_{21}$ (with e^{-ih} there replaced by q here). $A(R_{CG})$ again becomes a Hopf algebra, with the identification of the central quantum determinant found in [**52**], and will then be denoted $\mathbb{C}_{CG,q}[SL(n)]$.

The Fronsdal-Galindo R-matrix will be considered in the next section. However we will present it here so that the reader might easily compare it with the Cremmer-Gervais R-matrix. Thus, we take the Fronsdal-Galindo R-matrix to be

$$(R_{FG})^{st}_{ij} = \begin{cases} q & i = j = s = t, \\ q & i = s = 2N - j,\ j = t,\ 0 < j < N, \\ q^{-1} & i = s,\ j = t = 2N - i,\ 0 < i < N, \\ 1 & i = s,\ j = t,\ i \neq j,\ i + j \neq 2N, \\ q - q^{-1} & i = t < j = s, \\ q\kappa_i & 0 < i < N,\ j = 2N - i,\ s = t = N, \\ q\tilde{\kappa}_j & 0 < j < N,\ i = 2N - j,\ s = t = N, \\ q^{-1}\xi_{is} & 0 < i < s < N,\ j = 2N - i,\ t = 2N - s, \\ q\tilde{\xi}_{jt} & 0 < j < t < N,\ i = 2N - j,\ s = 2N - t. \end{cases} \tag{3.2.34}$$

where

$$\tilde{\kappa}_i = -q^{2(N-i)}\kappa_i, \tag{3.2.35}$$

$$\tilde{\xi}_{ij} = (1-q^{-2})q^{2(j-i)}(\kappa_i/\kappa_j), \tag{3.2.36}$$

$$\xi_{ij} = (1-q^2)(\kappa_i/\kappa_j). \tag{3.2.37}$$

Clearly, R_{FG} depends on N parameters — q together with κ_i, $0 < i < N$. In this case, if the R-matrix given in section 5 of the paper [**41**] (with the q there replaced by q^{-1}) is denoted $\tilde{R}$, then $R_{FG} = q\tilde{R}_{21}$. It will be shown that this R-matrix is related via twisting to R_S. Thus we can say that $A(R_{FG})$ may also be taken to be a Hopf algebra, which is more than is claimed in [**41, 42**]. We will denote this quantum group by $\mathbb{C}_{FG,q}[GL(2N-1)]$.

EXAMPLE 3.2.11. For the standard deformation $\mathbb{C}_q[SL(n)]$, defined by the R-form $\sigma_S(T_i^s, T_j^t) = (R_S)_{ij}^{st}$, we define the 2-cocycle χ by

$$\chi(T_i^s, T_j^t) = F_{ij}^{st} = f_{ij}\delta_i^s\delta_j^t, \tag{3.2.38}$$

extended to the whole of $\mathbb{C}_q[SL(n)]$ by (3.2.22) and (3.2.23). Note that the summation convention is not being assumed here and indeed will not be assumed anywhere, unless stated otherwise. We find that the conditions (3.2.24) and (3.2.25) impose no restrictions on the f_{ij}s. The new, twisted R-form then coming from (3.2.17) is

$$\sigma_{S,p}(T_i^s, T_j^t) = (R_{S,p})_{ij}^{st}, \tag{3.2.39}$$

where $R_{S,p}$ is the familiar $(1+\binom{n}{2})$-parameter standard R-matrix given by

$$(R_{S,p})_{ij}^{st} = \begin{cases} q & i=j=s=t, \\ p_{ij} & i=s\neq j=t, \\ (q-q^{-1}) & i=t<j=s, \end{cases} \tag{3.2.40}$$

with $p_{ij} = f_{ji}f_{ij}^{-1} = p_{ji}^{-1}$ for $i<j$. The multiparameter Hopf algebra so defined will be denoted $\mathbb{C}_{q,p}[GL(n)]$, and was first constructed in this way by Kempf in [**61**] (see also the paper by Schirrmacher [**85**]). We will often take $p_{ii} = q$ in what follows.

EXAMPLE 3.2.12. For the 1-parameter Cremmer-Gervais deformation $\mathbb{C}_{CG,q}[SL(n)]$ the situation is more interesting. We again define a 2-cocycle χ in terms of a diagonal matrix $F_{ij}^{st} = f_{ij}\delta_i^s\delta_j^t$. However now the compatibility conditions (3.2.24) and (3.2.25) *do* impose restrictions on the f_{ij}s. The number of independent parameters appearing in the twisted Hopf algebra $\mathbb{C}_{CG,q,p}[GL(n)]$ is then determined by the number of independent combinations of the f_{ij}s which appear in $R_{CG,p} = F_{21}R_{CG}F^{-1}$. As demonstrated in Appendix A, we are left with just three independent parameters — q together with a new pair, p and λ. Explicitly, the *3-parameter generalised Cremmer-Gervais R-matrix* is given by

$$(R_{CG,p})_{ij}^{st} = \begin{cases} q & i=j=s=t, \\ p^{j-s}q & i=s<j=t, \\ p^{j-s}q^{-1} & i=s>j=t, \\ (q-q^{-1}) & i=t<j=s, \\ p^{j-s}\lambda^{st-ij}(q-q^{-1}) & i<s<j, \text{ and } t=i+j-s, \\ -p^{j-s}\lambda^{st-ij}(q-q^{-1}) & j<s<i, \text{ and } t=i+j-s. \end{cases} \tag{3.2.41}$$

We refer the reader to Appendix A for the proof of this result.

EXAMPLE 3.2.13. Another type of Reshetikhin twist, is the sub-Hopf-algebra-induced twist, studied in particular by Engeldinger and Kempf [**37**]. An example of such a twist is given by defining a 2-cocycle on $\mathbb{C}_{q,p}[GL(n)]$ as

$$\chi(T_i^s, T_j^t) = \begin{cases} f_{ij} & i = s,\, j = t, \\ q^{-1}(q - q^{-1}) f_{\eta\eta} & i = t = \eta,\, j = s = \eta + 1, \end{cases} \tag{3.2.42}$$

with the following restrictions on the f_{ij}s to ensure that all the conditions of the twisting system are satisfied

$$f_{\eta\eta} = f_{\eta+1,\eta+1}, \tag{3.2.43}$$

$$f_{\eta,\eta+1} = q^{-1} p_{\eta,\eta+1} f_{\eta\eta}, \tag{3.2.44}$$

$$f_{\eta+1,\eta} = q^{-1} p_{\eta+1,\eta} f_{\eta\eta}, \tag{3.2.45}$$

$$f_{i,\eta+1} = p_{i,\eta+1} p_{\eta,i} f_{i\eta}, \quad i \neq \eta, \eta + 1, \tag{3.2.46}$$

$$f_{\eta+1,i} = p_{\eta+1,i} p_{i,\eta} f_{\eta i}, \quad i \neq \eta, \eta + 1. \tag{3.2.47}$$

The new R-form is then given by the R-matrix

$$(R_{EK})_{ij}^{st} = \begin{cases} q & i = j = s = t, \\ \tilde{p}_{ij} & i = s \neq j = t, \\ q - q^{-1} & i = t < j = s, \\ -(q - q^{-1}) & i = t = \eta,\, j = s = \eta + 1, \\ q - q^{-1} & i = t = \eta + 1,\, j = s = \eta, \end{cases} \tag{3.2.48}$$

where $\tilde{p}_{ij} = p_{ij} f_{ji} f_{ij}^{-1}$. There is no change in the number of independent parameters. The twist quoted here actually corresponds to an embedding of $U_q(\mathfrak{gl}_2(\mathbb{C}))$ in $U_q(\mathfrak{gl}_n(\mathbb{C}))$. There are many others, and we refer the reader to [**37**] for details.

3.3. A new class of twisting 2-cocycles

Our major results all appear as particular examples of a new twisting system, quite distinct from that of Reshetikhin, described in the following theorem.

THEOREM 3.3.1. *Suppose $A(R)$ is any FRT bialgebra, defined in terms of an $n \times n$ R-matrix R. To any $n \times n$ matrix F which satisfies the following conditions:*

$$F_{12} F_{23} = F_{23} F_{12}, \tag{3.3.1}$$

$$R_{12} F_{23} F_{13} = F_{13} F_{23} R_{12}, \tag{3.3.2}$$

$$R_{23} F_{12} F_{13} = F_{13} F_{12} R_{23}, \tag{3.3.3}$$

there corresponds a counital 2-cocycle χ defined on $A(R)$. It is given on the generators of $A(R)$ by

$$\chi(1, T) = \chi(T, 1) = \epsilon(T), \tag{3.3.4}$$

$$\chi(T_1, T_2) = F_{12}, \tag{3.3.5}$$

and extended to the whole algebra as

$$\chi(T_1 T_2, T_3) = \chi(T_2, T_3)\chi(T_1, T_3) = F_{23} F_{13}, \tag{3.3.6}$$

$$\chi(T_1, T_2 T_3) = \chi(T_1, T_2)\chi(T_1, T_3) = F_{12} F_{13}. \tag{3.3.7}$$

PROOF. The fact that χ is consistent with the underlying algebraic structure of $A(R)$ follows from (3.3.2) and (3.3.3), while the defining condition (3.2.13) follows easily on using (3.3.1) together with (3.3.6), (3.3.7) and the fact that $F_{ij}F_{kl} = F_{kl}F_{ij}$ whenever i, j, k and l are mutually distinct. $\square$

REMARK 3.3.2. There is of course a dual result to this, which applies to any quasitriangular Hopf algebra $(H, \mathcal{R})$: Given an invertible element $\mathcal{F} \in H \otimes H$ satisfying

$$\mathcal{F}_{12}\mathcal{F}_{23} = \mathcal{F}_{23}\mathcal{F}_{12}, \tag{3.3.8}$$

together with

$$(\Delta \otimes \mathrm{id})(\mathcal{F}) = \mathcal{F}_{23}\mathcal{F}_{13}, \tag{3.3.9}$$

and

$$(\mathrm{id} \otimes \Delta)(\mathcal{F}) = \mathcal{F}_{12}\mathcal{F}_{13}, \tag{3.3.10}$$

then $(H, \mathcal{R}_{\mathcal{F}})$ is a new quasitriangular Hopf algebra, with the coproduct, universal R-matrix and antipode twisted as in (3.2.29), (3.2.30) and (3.2.31) respectively.

Some of the general features of twists coming from this construction will be explicated in the following example.

EXAMPLE 3.3.3. Let us take as our initial object, the multiparameter standard quantum group $\mathbb{C}_{q,p}[GL(3)]$, and consider the possibility of defining on it a 2-cocycle χ defined on the generators as

$$\chi(T_i^s, T_j^t) = F_{ij}^{st} = \begin{cases} f_{ij} & i = s,\ j = t, \\ \mu & i = 1,\ j = 3,\ s = t = 2. \end{cases} \tag{3.3.11}$$

For F to satisfy (3.3.1) we need $f_{i1} = f_{i2}$ and $f_{2i} = f_{3i}$ for all $i = 1, \ldots, 3$. For χ to be compatible with the algebra structure of $\mathbb{C}_{q,p}[GL(3)]$ we need (3.3.2) and (3.3.3) to be satisfied, which further requires $p_{i2}f_{i3} = p_{i1}f_{i2}$ and $p_{3i}f_{2i} = p_{2i}f_{1i}$ where $p_{ii} = q$, for $i = 1, \ldots, 3$. For generic p_{ij} these equations have no solution. However, giving up a degree of freedom from the parameter space of $\mathbb{C}_{q,p}[GL(3)]$ by setting $p_{13} = qp_{12}p_{23}$, they can be solved. As a matrix, F is then given by

$$F = \begin{pmatrix} q^{-1}p_{32}f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^{-1}p_{32}f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_{21}p_{32}f & 0 & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{-1}p_{21}f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1}p_{21}f \end{pmatrix}, \tag{3.3.12}$$

where $f = f_{22}$. The R-matrix of $\mathbb{C}_{q,p}[SL(3)]$ with $p_{13} = qp_{12}p_{23}$ then twists to R_χ, where

$$R_\chi = \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & qp & 0 & q-q^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & qp^2 & 0 & -p^2qf^{-1}\mu & 0 & q-q^{-1} & 0 & 0 \\ 0 & 0 & 0 & q^{-1}p^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & qp & 0 & q-q^{-1} & 0 \\ 0 & 0 & 0 & 0 & qf^{-1}\mu & 0 & q^{-1}p^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1}p^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}, \tag{3.3.13}$$

with $p = p_{12}p_{23}$. It is clear that on choosing $f = -p\lambda^{-1}$ and $\mu = q^{-1}(q - q^{-1})$, we have obtained precisely the R-matrix $R_{CG,p}$ for $n = 3$.

The 2-cocycle here is an example of a general class of *simple root 2-cocycles* which are defined on $\mathbb{C}_{q,p}[GL(n)]$ for any pair of integers (k, l) such that $0 < k < l < n$ by

$$\chi(T_i^s, T_j^t) = F_{ij}^{st} = \begin{cases} f_{ij} & i = s,\ j = t, \\ \mu & i = k,\ j = l+1,\ s = k+1,\ t = l, \end{cases} \tag{3.3.14}$$

with the constraints

$$f_{i,k} = f_{i,k+1}, \qquad f_{l,i} = f_{l+1,i}, \tag{3.3.15}$$

and

$$p_{i,k}f_{i,l} = p_{i,k+1}f_{i,l+1}, \tag{3.3.16}$$

$$p_{l,i}f_{k,i} = p_{l+1,i}f_{k+1,i}, \tag{3.3.17}$$

for all $i = 1, \ldots, n$. The name comes from the fact that these twists add non-zero elements to the R-matrix at points corresponding to the the non-zero elements of the matrices $\Gamma(e_{\alpha_k}) \otimes \Gamma(e_{-\alpha_l})$, and $\Gamma(e_{-\alpha_l}) \otimes \Gamma(e_{\alpha_k})$, where the e_{α_k} are the basis elements corresponding to the simple roots of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$, and Γ is the first fundamental representation. These simple root 2-cocycles may be combined in more general *composite simple root 2-cocycles* defined on $\mathbb{C}_{q,p}[GL(n)]$ for each $0 < k < n$, by

$$\chi(T_i^s, T_j^t) = F_{ij}^{st} = \begin{cases} f_{ij} & i = s,\ j = t, \\ \mu_m & i = k,\ j = m+1,\ s = k+1,\ t = m. \end{cases} \tag{3.3.18}$$

With the constraints as before, and m now taking all possible values such that $k < m < n$, this twist imparts a whole series of non-standard off-diagonal elements to the R-matrix. Demonstration of the truth of these statements involves a straightforward verification of the conditions (3.3.1), (3.3.2) and (3.3.3).

While working on the universal T-matrix, Fronsdal and Galindo [**41, 42**] found an interesting non-standard deformation of $\mathbb{C}[GL(2N-1)]$, which we shall denote by $\mathbb{C}_{FG,q}[GL(2N-1)]$, and which they called 'esoteric'. We have already introduced their R-matrix in (3.2.34). For $N = 2$ this is precisely the generalised Cremmer-Gervais quantum group $\mathbb{C}_{CG,q,p}[GL(3)]$ with $p = q^{-1}$ and $\lambda = q^2(\kappa_1/(q - q^{-1}))$. However for $N > 2$, their quantum groups do not coincide with those of the Cremmer-Gervais series (c.f. added note in [**42**]). In fact, in a sense which we will make more precise in the next section, the quantum groups $\mathbb{C}_{FG,q}[GL(2N-1)]$ are 'not as non-standard' as those of Cremmer and Gervais. As we shall discuss later, the general Cremmer-Gervais quantum group does not seem to be a twisting of a standard quantum group by a 2-cocycle of the type we are considering here. However the quantum groups of Fronsdal and Galindo *are* obtained from standard-type quantum groups through a 2-cocycle which fits into our general scheme, and is presented in the following proposition.

PROPOSITION 3.3.4. *On the quantum group $\mathbb{C}_{q,p}[GL(2N-1)]$, with the parameters constrained according to*

$$p_{ji'} = qp_{jN}p_{Ni'}, \qquad \frac{p_{ij}}{p_{iN}p_{Nj}} = \frac{p_{i'j'}}{p_{i'N}p_{Nj'}}, \tag{3.3.19}$$

for all $0 < i, j < N$ and where $i' = 2N - i$, there is a 2-cocycle χ_{FG} defined as

$$\chi_{FG}(T_i^s, T_j^t) = F_{ij}^{st} = \begin{cases} f_{ij} & i = s,\ j = t, \\ \mu_k & i = k,\ j = k',\ s = N,\ t = N, \\ \lambda_{kl} & i = k,\ j = k',\ s = l,\ t = l', \end{cases} \tag{3.3.20}$$

where $0 < k < l < N$. All the f_{ij} are given in terms of f_{NN} according to

$$f_{ij} = \begin{cases} q^{-1}p_{i'N}f_{NN} & 0 < i, j \leq N, \\ p_{i'j}p_{jj'}f_{NN} & 0 < i \leq N < j < 2N, \\ f_{NN} & 0 < j \leq N < i < 2N, \\ q^{-1}p_{Nj'}f_{NN} & N < i, j < 2N. \end{cases} \tag{3.3.21}$$

The λs are determined in terms of the μs by

$$\lambda_{ij} = p_{j'j}f_{NN}(q - q^{-1})(\mu_i/\mu_j), \tag{3.3.22}$$

for all $0 < i < j < N$.

PROOF. This result is obtained by a applying conditions (3.3.1), (3.3.2) and (3.3.3), with $R = R_{S,p}$, to (3.3.20). $\square$

REMARK 3.3.5. It is not difficult to establish that the number of parameters left in $\mathbb{C}_{q,p}[GL(2N-1)]$ after imposing the conditions (3.3.19), is $(1 + \frac{1}{2}(N-1)(N+2))$. This is just the number of independent p_{ij}s.

Using the 2-cocycle (3.3.20) we in fact obtain an R-matrix more general than that of Fronsdal and Galindo. Details are given in Appendix B, where we obtain this *multi-parameter generalised Fronsdal-Galindo R-matrix* and demonstrate explicitly that their original R-matrix (3.2.34) is a special case of the new generalised R-matrix.

3.4. The Cremmer-Gervais problem for $GL(4)$ and beyond

In the semiclassical theory of quasitriangular Lie bialgebras associated with Lie algebras $\mathfrak{g}$, and their corresponding Poisson Lie groups (see for example the treatment in the book by Chari and Pressley [**17**]), the fundamental role is played by the classical r-matrix, $r \in \mathfrak{g} \otimes \mathfrak{g}$, which completely specifies the Lie bialgebra. In the case of complex, finite dimensional, simple Lie algebras, there is a complete classification of all such r-matrices, due to Belavin and Drinfeld [**10, 17**], in terms of 'admissible' or 'Belavin-Drinfeld' triples, (Π_1, Π_0, τ), where Π is the set of simple roots of $\mathfrak{g}$, Π_1, $\Pi_0 \subset \Pi$ and $\tau : \Pi_1 \to \Pi_0$ is a bijection. Considering in particular the situation for $\mathfrak{g} = \mathfrak{sl}_{l+1}(\mathbb{C})$, we can distinguish three cases of interest to us. In the standard, or Drinfeld-Jimbo case, the Belavin-Drinfeld triple has Π_1 and Π_0 both empty and the corresponding r-matrix, r_S, coincides with the semiclassical limit of the universal R-matrix, $\mathcal{R}_S$, of the familiar quasitriangular quantised universal enveloping algebra, $U_h(\mathfrak{sl}_{l+1}(\mathbb{C}))$. Another r-matrix, this time for $\mathfrak{sl}_{2N-1}(\mathbb{C})$, has **[50]** $\Pi_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_{N-1}\}$ and $\Pi_0 = \{\alpha_N, \alpha_{N+1}, \ldots, \alpha_{2(N-1)}\}$, where $\alpha_1, \ldots, \alpha_{2(N-1)}$ are the simple roots of $\mathfrak{sl}_{2N-1}(\mathbb{C})$. When considered in the first fundamental representation of $\mathfrak{sl}_{2N-1}(\mathbb{C})$, this can be seen to correspond to the semiclassical limit of a Fronsdal-Galindo type R-matrix (3.2.34), and so will be denoted r_{FG}. Finally, we have a r-matrix for $\mathfrak{sl}_{l+1}(\mathbb{C})$, in which Π_1 and Π_0 are as full as possible **[8]**, with $\Pi_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-2}\}$ and $\Pi_0 = \{\alpha_2, \alpha_{n+1}, \ldots, \alpha_{n-1}\}$, where $\alpha_1, \ldots, \alpha_{n-1}$ are the simple roots of $\mathfrak{sl}_{l+1}(\mathbb{C})$. This time, when viewed in the first fundamental representation of $\mathfrak{sl}_{l+1}(\mathbb{C})$, we find a correspondence with a Cremmer-Gervais type R-matrix (3.2.33), and so we will write this r-matrix as r_{CG}. Let us note, that in each of these cases, the element t defined as $t = r_{12} + r_{21}$ is identical, and is in fact the Casimir element of $\mathfrak{sl}_{l+1}(\mathbb{C}) \otimes \mathfrak{sl}_{l+1}(\mathbb{C})$.

In a series of fundamental works **[33, 35, 36]**, Drinfeld proved that given any Lie algebra, $\mathfrak{g}$, together with a symmetric $\mathfrak{g}$-invariant element, t, there exists a quantisation of the universal enveloping algebra, $U(\mathfrak{g})$, as a *quasitriangular quasi-Hopf quantised universal enveloping algebra*, $(U(\mathfrak{g})[[h]], \Phi, e^{ht/2})$, and that this quantisation is *unique up to twisting.* An immediate consequence of this result is that the standard quantisation, $(U_h(\mathfrak{sl}_{l+1}(\mathbb{C})), \mathcal{R}_S)$ of $U(\mathfrak{sl}_{l+1}(\mathbb{C}))$, is twist equivalent, as a *quasitriangular quasi-Hopf algebra*, to the 'universal' quantisation $(U(\mathfrak{sl}_{l+1}(\mathbb{C}))[[h]], \Phi, e^{ht/2})$. Subsequently **[34]**, Drinfeld formulated a number of unsolved problems in quantum group theory. Among these, was the question of whether *every* finite dimensional Lie bialgebra admits a quantisation as a quantised universal enveloping algebra. This was recently answered, in the affirmative, by Etingof and Kazhdan **[38]**. Though their result did not provide an explicit construction, it does tell us that in addition to the well known Drinfeld-Jimbo quasitriangular quantised universal enveloping algebra, $(U_h(\mathfrak{sl}_{l+1}(\mathbb{C})), \mathcal{R}_S)$, we must assume the *existence* of quasitriangular Fronsdal-Galindo and Cremmer-Gervais quantised universal enveloping algebras, with corresponding universal R-matrices $\mathcal{R}_{FG}$ and $\mathcal{R}_{CG}$ respectively. Moreover, by Drinfeld's earlier result, we know that these quantised universal enveloping algebras must be twist equivalent as *quasitriangular Hopf algebras*, to $(U_h(\mathfrak{sl}_{l+1}(\mathbb{C})), \mathcal{R}_S)$. In particular, the universal R-matrices, $\mathcal{R}_S$, $\mathcal{R}_{FG}$, and $\mathcal{R}_{CG}$, must each be related to each other by twisting in the style of equation (3.2.30).

It is reasonable, we believe, to work under the motivating assumption that the matrices R_{FG} and R_{CG}, which have been considered in this chapter, correspond to the, as yet unknown, universal R-matrices, $\mathcal{R}_{FG}$ and $\mathcal{R}_{CG}$, in the first fundamental representation. In this case, the theory we have just outlined implies that in the dual world of co-quasitriangular Hopf algebras, there should exist 2-cocycles for the construction of the Fronsdal-Galindo quantum groups and the Cremmer-Gervais quantum groups from the standard quantum groups. Some support for the assumption has been provided in this chapter, with the explicit construction of a twisting 2-cocycle for the construction of the Fronsdal-Galindo quantum groups, and the Cremmer-Gervais deformation of $GL(3)$. However the problem for the Cremmer-Gervais deformations of $GL(n)$ for $n > 3$ remains open.

The pair of non-standard off-diagonal elements which appear in the Cremmer-Gervais R-matrix for $GL(3)$ 'correspond' in the sense described above, to the element $e_{\alpha_1} \wedge e_{-\alpha_2}$ of the corresponding classical r-matrix, r_{CG}. As we have seen, our twisting construction has no problem dealing with this case. However, for $GL(4)$ and beyond, the Cremmer-Gervais R-matrix involves an increasing number of non-simple root combinations — more than appear in the Fronsdal-Galindo R-matrix, and our construction does not appear to be able to deal with this circumstance. In particular, in the Cremmer-Gervais R-matrix for $GL(4)$, there are pairs of non-standard matrix elements corresponding to the r-matrix elements $e_{\alpha_1} \wedge e_{-\alpha_2}$, $e_{\alpha_1} \wedge e_{-\alpha_3}$, $e_{\alpha_2} \wedge e_{-\alpha_3}$ and $e_{\alpha_1+\alpha_2} \wedge e_{-(\alpha_2+\alpha_3)}$. The last term, in particular, causes problems. We finish by explaining how a *new* non-standard $GL(4)$ R-matrix may be obtained, which contains a pair of matrix elements corresponding to this term.

Starting from the standard quantum group $\mathbb{C}_{q,p}[GL(4)]$, and twisting first using a 2-cocycle of the kind in (3.2.42), we obtain a new quantum group given in terms of the R-matrix

$$(R_{EK})_{ij}^{st} = \begin{cases} q & i=j=s=t, \\ \tilde{p}_{ij} & i=s\neq j=t, \\ q-q^{-1} & i=t<j=s, \\ -(q-q^{-1}) & i=t=2,\, j=s=3, \\ q-q^{-1} & i=t=3,\, j=s=2. \end{cases} \tag{3.4.1}$$

This quantum group is now amenable to a twist by one of our new 2-cocycles, given by

$$\chi(T_i^s, T_j^t) = F_{ij}^{st} = \begin{cases} f_{ij} & i=s,\, j=t, \\ \lambda & i=1,\, j=4,\, s=3,\, t=2, \end{cases} \tag{3.4.2}$$

with the constraints

$$f_{i1} = f_{i3}, \qquad f_{2i} = f_{4i}, \tag{3.4.3}$$

$$\tilde{p}_{i1} f_{i2} = \tilde{p}_{i3} f_{i4}, \qquad \tilde{p}_{4i} f_{3i} = \tilde{p}_{2i} f_{1i}, \tag{3.4.4}$$

for $i = 1, \ldots, 4$. Note that the this 2-cocycle could not have been defined on the original standard R-matrix. The R-matrix for this *new* non-standard quantum group may now be written as

$$(R_{NS})_{ij}^{st} = \begin{cases} q & i=j=s=t, \\ \gamma_{ij} & i=s\neq j=t, \\ q-q^{-1} & i=t<j=s, \\ -(q-q^{-1}) & i=t=2,\, j=s=3, \\ q-q^{-1} & i=t=3,\, j=s=2, \\ \gamma_{14}\varrho & i=1,\, j=4,\, s=3,\, t=2, \\ -q\gamma_{23}\varrho & i=4,\, j=1,\, s=2,\, t=3, \end{cases} \tag{3.4.5}$$

where

$$\gamma_{ij} = \tilde{p}_{ij} f_{ji} f_{ij}{}^{-1}, \qquad \varrho = -\lambda f_{14}{}^{-1} f_{32}{}^{-1}, \tag{3.4.6}$$

and

$$\gamma_{12}\gamma_{23} = q\gamma_{24}, \qquad \gamma_{24}\gamma_{34} = q\gamma_{14}. \tag{3.4.7}$$

This new R-matrix depends on 6 parameters. It might be interesting to investigate such double twists further.

3.5. Appendix A: The 3-parameter generalised Cremmer-Gervais R-matrix

We give here the derivation of the result quoted in Example 3.2.12. We proceed in two stages, obtaining the required result by demonstrating that the R-matrix (3.2.41) is a twist of the R-matrix (3.2.33).

1. As explained in Section 2, as any diagonal matrix $F^{kl}_{ij} = f_{ij}\delta^k_i\delta^l_j$ is a solution of the QYBE, we can define a twisting 2-cocycle χ in terms of it as $\chi(T_1, T_2) = F_{12}$ as long as the compatibility conditions (3.2.24) and (3.2.25) are satisfied. In terms of matrix components, these conditions become

$$R^{st}_{ij} f_{s\alpha} f_{t\alpha} = f_{j\alpha} f_{i\alpha} R^{st}_{ij}, \tag{3.5.1}$$

and

$$R^{st}_{ij} f_{\alpha t} f_{\alpha s} = f_{\alpha i} f_{\alpha j} R^{st}_{ij}, \tag{3.5.2}$$

$i, j, \alpha, s, t = 1, \ldots, n$, $n \geq 3$. Thus the non-zero elements of the R-matrix R_{CG} determine the constraints on the elements of F. It is not difficult to see that the only non-trivial relations which we get are

$$f_{i\alpha} f_{j\alpha} = f_{s\alpha} f_{t\alpha} \quad i < s < j, \;\; t = i + j - s, \tag{3.5.3}$$

and

$$f_{\alpha i} f_{\alpha j} = f_{\alpha s} f_{\alpha t} \quad i < s < j, \;\; t = i + j - s, \tag{3.5.4}$$

$i, j, \alpha, s, t = 1, \ldots, n$, $n \geq 3$. We will now prove the following lemma.

LEMMA 2. *The system of equations (3.5.3) and (3.5.4), in n^2 unknowns, has a solution space completely described in terms of four unknowns x, y, z, w say, as*

$$f_{ij} = x^{(i-2)(j-2)} y^{-(i-2)(j-1)} z^{-(i-1)(j-2)} w^{(i-1)(j-1)}, \tag{3.5.5}$$

$i, j = 1, \ldots, n$.

PROOF. We use induction. Consider the simplest case, $n = 3$, so that we have $i = 1$, $s = t = 2$, $j = 3$ and there are six equations

$$f_{11} f_{31} = {f_{21}}^2, \tag{3.5.6}$$

$$f_{12} f_{32} = {f_{22}}^2, \tag{3.5.7}$$

$$f_{13} f_{33} = {f_{23}}^2, \tag{3.5.8}$$

$$f_{11} f_{13} = {f_{12}}^2, \tag{3.5.9}$$

$$f_{21} f_{23} = {f_{22}}^2, \tag{3.5.10}$$

$$f_{31} f_{33} = {f_{32}}^2. \tag{3.5.11}$$

Only five of these are independent, e.g. combining (3.5.6), (3.5.7), (3.5.8), (3.5.9) and (3.5.10) yields (3.5.11), so the solution space will be in terms of four unknowns. Choosing these to be $f_{11} = x$, $f_{12} = y$, $f_{21} = z$ and $f_{22} = w$, we find

$$\|f_{ij}\| = \begin{pmatrix} x & y & x^{-1}y^2 \\ z & w & z^{-1}w^2 \\ x^{-1}z^2 & y^{-1}w^2 & xy^{-2}z^{-2}w^4 \end{pmatrix}, \tag{3.5.12}$$

which verifies (3.5.5) for $n = 3$. Now suppose that the solution space of the system of equations (3.5.3) and (3.5.4) for $n = k$, $k \geq 3$ is completely specified by (3.5.5), and consider $n = k + 1$. Notice that we still have all the equations we had for $n = k$ so (3.5.5) holds for $i, j = 1, \ldots, k$. We need to check that the new equations appearing consistently specify $f_{\alpha,k+1}$ and $f_{k+1,\alpha}$ according to (3.5.5) for $\alpha = 1, \ldots, k + 1$.

From (3.5.4), for $\alpha = 1, \ldots, k$,

$$\begin{aligned} f_{\alpha,k+1} &= {f_{\alpha i}}^{-1} f_{\alpha s} f_{\alpha t}, \\ &= x^{-(\alpha-2)(i-2)+(\alpha-2)(s-2)+(\alpha-2)(t-2)} \\ &\times y^{(\alpha-2)(i-1)-(\alpha-2)(s-1)-(\alpha-2)(t-1)} \\ &\times z^{(\alpha-1)(i-2)-(\alpha-1)(s-2)-(\alpha-1)(t-2)} \\ &\times w^{-(\alpha-1)(i-1)+(\alpha-1)(s-1)+(\alpha-1)(t-1)}, \\ &= x^{(\alpha-2)(-i+s+t-2)} y^{(\alpha-2)(i-s-t+1)} z^{(\alpha-1)(i-s-t+2)} w^{(\alpha-1)(-i+s+t-1)}, \\ &= x^{(\alpha-2)(k+1-2)} y^{-(\alpha-2)(k+1-1)} z^{-(\alpha-1)(k+1-2)} w^{(\alpha-1)(k+1-1)}, \end{aligned}$$

as required. Invoking the obvious symmetry between (3.5.3) and (3.5.4) we deduce the equivalent result from (3.5.3) for $f_{k+1,\alpha}$, $\alpha = 1, \ldots, k$. Replacing α by $k + 1$ in the above computation yields the correct result for $f_{k+1,k+1}$. The consistency of these solutions still needs to be checked, but follows from the following. Take $f_{\alpha,k+1} = f_{\alpha s} f_{\alpha t} {f_{\alpha i}}^{-1}$ from (3.5.4), and consider (3.5.3) with $\alpha = k + 1$, i.e.

$$f_{i',k+1} f_{j',k+1} = f_{s',k+1} f_{t',k+1},$$

where $i' < s' < j'$, and $t' = i' + j' - s'$, $i', s', t', j' = 1, \ldots, k + 1$. Then

$$\begin{aligned} \text{LHS} &= f_{i's} f_{i't} {f_{i'i}}^{-1} f_{j's} f_{j't} {f_{j'i}}^{-1} \\ &= f_{i's} f_{i't} f_{j'i} {f_{s'i}}^{-1} {f_{t'i}}^{-1} f_{j's} f_{j't} {f_{j'i}}^{-1} \\ &= f_{s's} f_{t's} f_{s't} f_{t't} {f_{s'i}}^{-1} {f_{t'i}}^{-1} \\ &= f_{s',k+1} f_{t',k+1} \\ &= \text{RHS} \end{aligned}$$

□

2. We must now consider what combinations of the f_{ij}s actually take part in the twisting. Thus, we consider the matrix $R_{CG,p} = F_{21} R_{CG} F^{-1}$, whose components are given by $(R_{CG,p})_{ij}^{st} = f_{ji}(R_{CG})_{ij}^{st}{f_{st}}^{-1}$. Explicitly

$$(R_{CG,p})_{ij}^{st} = \begin{cases} q & i=j=s=t, \\ f_{ji}{f_{ij}}^{-1}qq^{-2(j-s)/n} & i=s<j=t, \\ f_{ji}{f_{ij}}^{-1}q^{-1}q^{-2(j-s)/n} & i=s>j=t, \\ (q-q^{-1}) & i=t<j=s, \\ f_{ji}{f_{st}}^{-1}(q-q^{-1})q^{-2(j-s)/n} & i<s<j, \text{ and } t=i+j-s, \\ -f_{ji}{f_{st}}^{-1}(q-q^{-1})q^{-2(j-s)/n} & j<s<i, \text{ and } t=i+j-s, \end{cases} \tag{3.5.13}$$

and we are led to define

$$q_{ij} = f_{ij}{f_{ji}}^{-1}q^{-2(i-j)/n}, \qquad i,j=1,\ldots,n, \tag{3.5.14}$$

$$\lambda_{ijst} = f_{ij}{f_{st}}^{-1}q^{-2(i-s)/n}, \qquad i<s<j \text{ or } j<s<i \text{ and } t=i+j-s. \tag{3.5.15}$$

The q_{ij} and λ_{ijst} satisfy the following obvious symmetries

$$q_{ji} = {q_{ij}}^{-1}, \qquad i,j=1,\ldots,n, \tag{3.5.16}$$

$$\lambda_{jist} = q_{ji}\lambda_{ijst}, \qquad i<s<j \text{ and } t=i+j-s, \tag{3.5.17}$$

$$\lambda_{ijts} = q_{st}\lambda_{ijst}, \qquad i<s<j \text{ and } t=i+j-s. \tag{3.5.18}$$

Moreover, we see that modulo these symmetries every λ_{ijst} must either be of the form $\lambda_{ij\alpha\alpha}$, when $i+j$ is even, or $\lambda_{ij\alpha,\alpha+1}$ when $i+j$ is odd *or* be expressible in terms of these as

$$\lambda_{ijst} = \begin{cases} \lambda_{ij\alpha\alpha}/\lambda_{st\alpha\alpha} & i+j \text{ even}, \\ \lambda_{ij\alpha,\alpha+1}/\lambda_{st\alpha,\alpha+1} & i+j \text{ odd}. \end{cases} \tag{3.5.19}$$

Now, recalling the solution (3.5.5), we find that that $q_{ij} = y^{-i+j}z^{i-j}q^{-2(i-j)/n}$, so that on defining $p = y^{-1}zq^{-2/n}$, we have that $q_{ij} = p^{i-j}$. Now consider the λ_{ijst}s. From (3.5.5) we get

$$\begin{aligned} \lambda_{ij\alpha\alpha} &= x^{-(\alpha-i)^2}y^{(\alpha-i)(\alpha-i+1)}z^{(\alpha-i)(\alpha-i-1)}w^{-(\alpha-i)^2}q^{-2(i-\alpha)/n} \\ &= (y^{-1}zq^{-2/n})^{(i-\alpha)}(x^{-1}yzw^{-1})^{(\alpha-i)^2} \\ &= p^{(i-\alpha)}(x^{-1}yzw^{-1})^{(\alpha-i)^2}, \end{aligned} \tag{3.5.20}$$

$$\begin{aligned} \lambda_{ij\alpha,\alpha+1} &= x^{-(\alpha-i)(\alpha-i+1)}y^{(\alpha-i)(\alpha-i+2)}z^{(\alpha-i)^2}w^{-(\alpha-i)(\alpha-i+1)}q^{-2(i-\alpha)/n} \\ &= (y^{-1}zq^{-2/n})^{(i-\alpha)}(x^{-1}yzw^{-1})^{(\alpha-i)(\alpha-i+1)} \\ &= p^{(i-\alpha)}(x^{-1}yzw^{-1})^{(\alpha-i)(\alpha-i+1)}, \end{aligned} \tag{3.5.21}$$

so that on defining $\lambda = x^{-1}yzw^{-1}$, and recalling (3.5.19), we find that

$$\lambda_{ijst} = p^{i-s}\lambda^{st-ij}, \qquad i<s<j \text{ and } t=i+j-s. \tag{3.5.22}$$

This completes the derivation.

3.6. Appendix B: The Multiparameter Generalised Fronsdal-Galindo R-matrix

In Proposition 3.3.4 we presented the 2-cocycle χ_{FG} defined on a certain standard quantum group in terms of a matrix F. To obtain the twisted quantum group we also need ${\chi_{FG}}^{-1}$, which is defined in terms of F^{-1}

$$(F^{-1})_{ij}^{st} = \begin{cases} {f_{ij}}^{-1} & i = s,\ j = t, \\ \bar{\mu}_k & i = k,\ j = k',\ s = N,\ t = N, \\ \bar{\lambda}_{kl} & i = k,\ j = k',\ s = l,\ t = l', \end{cases} \tag{3.6.1}$$

where $0 < k < N$, $0 < k < l < N$ and

$$\bar{\mu}_i = -qq^{i-i'} p_{ii'} {f_{NN}}^{-2} \mu_i, \quad 0 < i < N, \tag{3.6.2}$$

$$\bar{\lambda}_{ij} = -q^{2(i-j)} p_{ii'} p_{jj'} {f_{NN}}^{-2} \lambda_{ij}, \quad 0 < i < j < N. \tag{3.6.3}$$

Now we determine the twisted R-matrix, $R_{FG,p}$, from $R_{FG,p} = F_{21} R_{S,p} F^{-1}$ as

$$(R_{FG,p})_{ij}^{st} = \begin{cases} p_{ij} f_{ji} {f_{ij}}^{-1} & i = s,\ j = t, \\ \bar{\mu}_k f_{k'k} p_{kk'} & i = k,\ j = k',\ s = t = N, \\ \bar{\lambda}_{kl} f_{k'k} p_{kk'} & i = k,\ j = k',\ s = l,\ t = l', \\ \mu_k {f_{NN}}^{-1} p_{NN} & i = k',\ j = k,\ s = t = N, \\ \lambda_{kl} {f_{l'l}}^{-1} p_{l'l} & i = k',\ j = k,\ s = l',\ t = l, \\ (q - q^{-1}) & i = t < j = s, \end{cases} \tag{3.6.4}$$

where $0 < k < N$ and $0 < k < l < N$, the p_{ij}s and f_{ij}s are constrained according to (3.3.19) and (3.3.21) respectively, and all other parameters are given in terms of the μs. We can refine the presentation of this R-matrix, setting

$$p_i = p_{ii'}, \tag{3.6.5}$$

$$\eta_{ij} = p_{ij} p_{j'i}, \tag{3.6.6}$$

$$\kappa_k = q^{-1} f_{NN} \bar{\mu}_k, \tag{3.6.7}$$

$$\tilde{\kappa}_k = -q^{2(N-k)} \kappa_k, \tag{3.6.8}$$

$$\xi_{kl} = (1 - q^2)(\kappa_k / \kappa_l), \tag{3.6.9}$$

$$\tilde{\xi}_{kl} = (1 - q^{-2}) q^{2(l-k)} (\kappa_k / \kappa_l). \tag{3.6.10}$$

Then the R-matrix becomes the *multiparameter generalised Fronsdal-Galindo R-matrix*

$$(R_{FG,p})_{ij}^{st} = \begin{cases} q & i = j = s = t, \\ q{p_i}^2 & i = s = j',\ j = t,\ 0 < j < N, \\ q^{-1}{p_i}^2 & i = s,\ j = t = i',\ 0 < i < N, \\ p_{j'} & i = s = N,\ j = t \neq N, \\ p_i & i = s \neq N,\ j = t = N, \\ p_i \eta_{ij} & i = s \neq N,\ j = t \neq N,\ i \neq j,\ i + j \neq 2N, \\ q - q^{-1} & i = t < j = s, \\ q p_i \kappa_i & 0 < i < N,\ j = 2N - i,\ s = t = N, \\ q p_{j'} \tilde{\kappa}_j & 0 < j < N,\ i = 2N - j,\ s = t = N, \\ q^{-1} p_i p_s \xi_{is} & 0 < i < s < N,\ j = 2N - i,\ t = 2N - s, \\ q p_{j'} p_{t'} \tilde{\xi}_{jt} & 0 < j < t < N,\ i = 2N - j,\ s = 2N - t. \end{cases} \tag{3.6.11}$$

It can be checked that this R-matrix has $(1 + \frac{1}{2}(N-1)(N+2))$ parameters.

To identify the R-matrix originally discussed by Fronsdal and Galindo (3.2.34), as a special case of this R-matrix, consider the particular solution of (3.3.19) given by setting $p_{ij} = 1$ for $0 < i \neq j < N$, $0 < i < N < j < 2N$ and $N < i \neq j < 2N$.

CHAPTER 4

Classification of bicovariant differential calculi on the Jordanian quantum groups $GL_{h,g}(2)$ and $SL_h(2)$ and quantum Lie algebras

4.1. Introduction

The program of noncommutative geometry pioneered by Connes [**19, 20**] is based on fundamental results in the field of abstract analysis discovered in the first half of this century by Gelfand, Kolmogoroff, Naimark, Stone and others (a useful historical overview can be found in I. Segal's review [**90**] of Connes' book [**19**]). In particular, Gelfand and Kolmogoroff showed that for a locally compact space, the algebra of continuous functions on the space is essentially *equivalent* to the space itself. The algebra of continuous functions is of course commutative, and in fact a C^*-algebra. We can then reasonably consider the study of *noncommutative* C^*-algebras as some form of noncommutative geometry. Thus the essential idea is to express the formalism of classical geometry as far as possible in the language of commutative algebra, and then use this as the paradigm for generalising to the noncommutative setting.

An implementation of this program has been developed by Dubois-Violette and coworkers (see the book by Madore [**68**], and the references therein). They generalize an elegant algebraic approach to the differential geometry of a smooth manifold introduced by Koszul [**63**] to the case where the commutative algebra of smooth functions on the manifold is replaced by the noncommutative algebra of matrices over some field.

In another direction, quantum groups provide natural candidates for noncommutative generalizations of the algebras of smooth functions on classical compact Lie groups. Classically the algebra of *representative functions* is a dense subalgebra of the algebra of all smooth complex valued functions on the group and carries the structure of a $\star$-Hopf algebra. Dropping the $\star$-structure we obtain the coordinate rings of the corresponding complex Lie groups, $SL_n(\mathbb{C})$, $SO_n(\mathbb{C})$ and $Sp_{2n}(\mathbb{C})$. These coordinate rings are generated already by the matrix elements of the defining representations of these groups (for $GL_n(\mathbb{C})$ we should adjoin $(\det(\mathbf{t}))^{-1}$ where $\mathbf{t}$ is the matrix of matrix element functions). Corresponding to these classical groups are the well known FRT Hopf algebras $A(R)$, introduced by Faddeev, Reshetikhin and Takhtajan in [**84**]. They are 'quantisations' of the classical coordinate rings. It is these structures upon which we should try to develop some sort of noncommutative Lie group geometry. The pioneering works here are those of Woronowicz [**103, 104**]. In particular, in [**104**] Woronowicz set out a formalism in terms of *bicovariant bimodules* which has been studied intensively by very many authors since. Let us note that the classical differential calculus on Lie groups is bicovariant.

There is a well known 'problem' with the bicovariant calculi associated with the standard FRT quantum groups other than $GL_q(n)$: Their dimensions do not agree with the corresponding classical calculi. In the particular case of $SL_q(n)$ the bicovariant calculus of Woronowicz is n^2-dimensional while the classical calculus is of dimension n^2-1 (the dimension being the dimension of the vector space of left-invariant 1-forms). This problem has stimulated some authors to consider alternative approaches to the development of differential geometry on quantum groups. For example, in [**87**] Schmüdgen and Schüler consider left-covariant bimodules (developing Woronowicz's original approach [**103**]) on $SL_q(n)$. They obtain first order calculi with the classical dimension; however for $n \geq 4$ the higher order calculi do not have the correct dimension. Another interesting approach was initiated by Faddeev and Pyatov[**39**]. They considered, for the particular case of $SL_q(n)$, the consequences of relaxing the condition of the classical Leibniz rule which is present in the bicovariant Woronowicz approach. They obtained bicovariant calculi of the correct classical dimension at all orders. However, subsequent work by Arutyunov, Isaev and Popowicz[**5**] suggests that a similar approach cannot be employed for the other simple quantum groups $SO_q(n)$ and $Sp_q(n)$.

In this chapter we consider the original Woronowicz bicovariant calculus, but we examine such calculi on the *non-standard* quantum groups $GL_{g,h}(2)$ and $SL_h(2)$ — the so-called Jordanian quantum groups. In [**58, 59**], Karimipour initiated the study of bicovariant calculi associated with $SL_h(2)$. Here we perform a complete classification of all first order bicovariant calculi on the quantum groups $GL_{h,g}(2)$ and $SL_h(2)$. Furthermore we consider the higher order calculi and the corresponding quantum Lie algebras. Let us summarise our main classification results:

- There are three 1-parameter families of 4-dimensional first order bicovariant differential calculi on $GL_{h,g}(2)$ whose bimodules of forms are generated as left $GL_{h,g}(2)$-modules by the differentials of the quantum group generators.
- For one value of the parameter, the calculi in the three families are the same. This parameter value coincides with the value required for a 'classical-like' reduction to a 3-dimensional first order bicovariant calculus on $SL_h(2)$ which is shown to be unique.
- For all the calculi the relations in the exterior algebra are obtained and are shown to lead to exterior calculi whose dimension is classical at all orders.
- For all the calculi the relations in the enveloping algebra of the quantum Lie algebra are obtained and are shown to lead to PBW-type bases.

Classically, the Lie algebra of a Lie group is obtained as the vector space of tangent vectors at the identity equipped with a Lie bracket defined in terms of the left-invariant vector fields on the group manifold. The formalism of Woronowicz's bicovariant calculus has a natural construction for a 'quantum Lie algebra' generalising the classical construction to the abstract Hopf algebra setting. However, for all standard quantum groups the quantum Lie algebras so obtained have the 'wrong' dimension. This has prompted authors such as Sudbery and Delius to look for alternative constructions for quantum Lie algebras [**67, 23, 24**]. Two different approaches are described in their work. Recall that classically the Lie algebra $\mathfrak{g}$ is an ad-submodule of the classical adjoint $U(\mathfrak{g})$-module, $U(\mathfrak{g})$, and its Lie bracket is the restriction of this classical adjoint action to $\mathfrak{g}$. This motivates the first approach [**67, 23**], **Approach 1**, in which we look for an ad-submodule *within* the quantised universal enveloping algebra $U_q(\mathfrak{g})$ which has the correct dimension and upon which the restriction of the adjoint action of $U_q(\mathfrak{g})$ closes. Lyubashenko and Sudbery employed a result of Joseph and Letzter and obtained a quantum Lie algebra in the cases $U_q(\mathfrak{sl}_{l+1}(\mathbb{C}))$ such that the coproduct of $U_q(\mathfrak{sl}_{l+1}(\mathbb{C}))$ applied to the basis of their quantum Lie algebra is of a particularly neat form. The other approach[**24**], **Approach 2**, (see also the paper by Bremner [**12**] and the earlier paper of Donin and Gurevich [**31**] where the idea appeared originally) constructs a quantum Lie algebra independent of any embedding into $U_q(\mathfrak{g})$. The idea here is to recall that classically the Lie bracket is an intertwiner, $[,] : \mathrm{ad} \otimes \mathrm{ad} \to \mathrm{ad}$, where ad is the usual adjoint representation of $U(\mathfrak{g})$. The quantum Lie bracket is then obtained as the intertwiner of the corresponding $U_q(\mathfrak{g})$-modules. Furthermore, classically the Killing form is an intertwiner, $\mathfrak{B} : \mathrm{ad} \otimes \mathrm{ad} \to \mathbb{C}$, and considering the intertwiner between the tensor product of the two (quantum) adjoint modules and the trivial representation an analog of the Killing form for the quantum Lie algebras is obtained in a rather straightforward manner. The two approaches lead to isomorphic quantum Lie algebras. However each has advantages over the other: **Approach 1** allows us to see explicitly the relationship between the quantum Lie algebra and the quantised universal enveloping algebra and in principle allows us to determine the coproduct of $U_q(\mathfrak{g})$ on the quantum Lie algebra; **Approach 2** gives a reasonably simple prescription for constructing quantum Lie algebras based on computation of inverse Clebsch-Gordan coefficients [**25**] and also direct means of establishing the quantum Killing form.

In the last part of the chapter we pursue this line of enquiry starting with the Jordanian quantised enveloping algebra $U_h(\mathfrak{sl}_2(\mathbb{C}))$. The following results are obtained:

- A Jordanian quantum Lie algebra is obtained according to **Approach 1** and the expressions for the coproduct of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ on the basis elements of the quantum Lie algebra are obtained.
- Following **Approach 2** we obtain a quantum Killing form for the quantum Lie algebra.
- We observe that the quantum Lie algebras obtained equivalently by **Approach 1** and **Approach 2** are isomorphic to the quantum Lie algebra obtained through the unique bicovariant calculus on the Jordanian quantum group $SL_h(2)$.

The chapter is organised as follows: In Section 4.2 we recall pertinent definitions and results concerning the Jordanian quantum groups $GL_{h,g}(2)$ and $SL_h(2)$. In Section 4.3 Woronowicz's theory is reviewed in a style intended to enlighten the details of the classification procedure, due to Müller-Hoissen [**77**], which we present in Section 4.4. The classification results appear in Sections 4.5 and 4.6. The Jordanian quantised universal enveloping algebra is recalled in Section 4.7 with the corresponding quantum Lie algebras obtained in Sections 4.8 and 4.9 through **Approach 1** and **Approach 2** respectively. In the final Section 4.10 we complete the picture by observing that these quantum Lie algebras are isomorphic to the one obtained from the bicovariant calculus on the Jordanian quantum group $SL_h(2)$.

4.2. The Jordanian quantum groups

The 2-parameter Jordanian quantum group $GL_{h,g}(2)$ is the co-quasitriangular Hopf algebra derived from the following R-matrix,

$$R = \begin{pmatrix} 1 & -h & h & gh \\ 0 & 1 & 0 & -g \\ 0 & 0 & 1 & g \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.2.1}$$

Being triangular, i.e. $R_{21}R = I$, this R-matrix is trivially Hecke, with

$$(\hat{R} - 1)(\hat{R} + 1) = 0, \tag{4.2.2}$$

where $\hat{R} = PR$ and $P_{ij,kl} = \delta_{il}\delta_{jk}$.

The quantum group associated with the R-matrix $R_{21} = R(-h, -g)$, with $g = h = 1$, was first investigated by Demidov et al. [**26**], while R_{21} with $g = h$ is the 1-parameter non-standard R-matrix whose quantum group was considered by Zakrzewski [**105**]. Lazarev and Movshev[**66**] considered the quantum group associated with R with $g = h$ and also the corresponding quantised universal enveloping algebra, $U_h(\mathfrak{sl}_2(\mathbb{C}))$. The quantised enveloping algebra was also investigated by Ohn [**81**] and will be discussed in more detail in Section 4.7. In fact, the R-matrix, (4.2.1), can be extracted from an early work of Gurevich [**47**]; though the associated quantum group structure was not investigated there.

In the usual way, defining an algebra valued matrix T as

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{4.2.3}$$

the relations of a matrix element bialgebra $A(R)$ are obtained from the well-known FRT [**84**] matrix relation

$$RT_1T_2 = T_2T_1R, \tag{4.2.4}$$

as

$$\begin{aligned} ca &= ac - gc^2, \quad cd = dc - hc^2, \\ db &= bd + g(ad - bc + hac - d^2), \\ ab &= ba + h(ad - bc + hac - a^2), \\ cb &= bc - hac - gdc + ghc^2, \quad da = ad + hac - gdc. \end{aligned} \tag{4.2.5}$$

The coalgebra structure is provided by a coproduct defined on the generators as

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d, \\ \Delta(c) &= c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d, \end{aligned} \tag{4.2.6}$$

with the counit given by

$$\begin{aligned} \epsilon(a) &= 1, \quad \epsilon(b) = 0, \\ \epsilon(c) &= 0, \quad \epsilon(d) = 1. \end{aligned} \tag{4.2.7}$$

$\hat{R}$ has a spectral decomposition

$$\hat{R} = P^+ + P^-, \tag{4.2.8}$$

where $P^+ = \frac{1}{2}(\hat{R} + I)$ is a rank 3 projector and $P^- = \frac{1}{2}(\hat{R} - I)$ is a rank 1 projector. In the notation of Majid [**71**], these projectors provide the associated quantum co-plane, $\mathbb{A}^{2|0}_{-1}$, and plane, $\mathbb{A}^{2|0}_{1}$, respectively through the relations $P^{\pm}\mathbf{x}_1\mathbf{x}_2 = 0$ where $\mathbf{x}$ is the 2×1 column vector $\binom{x_1}{x_2}$. These are associative algebras generated by the elements x_1 and x_2 subject to the relations

$$x_2^2 = 0, \quad x_1^2 = -hx_2x_1, \quad x_1x_2 = -x_2x_1, \tag{4.2.9}$$

in the case of the co-plane, $\mathbb{A}^{2|0}_{-1}$, and

$$x_1x_2 = x_2x_1 + gx_2^2 \tag{4.2.10}$$

in the case of the plane $\mathbb{A}^{2|0}_{1}$. A result of Mukhin [**76**, Theorems 1,2,3] (see also [**27**, Theorem 3.5]) then tells us that $A(R)$ is the *universal coacting bialgebra* on this pair of algebras in the sense of Manin [**73**]. In the language of Sudbery [**96**], $\mathbb{A}^{2|0}_{-1}$ and $\mathbb{A}^{2|0}_{1}$ are then *complementary coordinate algebras* determining $A(R)$, and with an easy application of the Diamond Lemma [**11**] telling us that they are moreover *superpolynomial algebras* generated by odd and even generators respectively having ordering algorithms with respect to the ordering of the generators, $x_2 \prec x_1$, we deduce immediately from a result of Sudbery [**96**, Theorem 3] that $A(R)$ has as a basis the ordered monomials $\{b^\alpha a^\beta d^\gamma c^\delta : \alpha, \beta, \gamma, \delta \in \mathbb{Z}_{\geq 0}\}$. This fact is used extensively in the computations which lead to our main results.

It follows from $\mathbb{A}^{2|0}_{-1}$, that R is *Frobenius* [**17**] and so in the usual way we can obtain a group-like element in the bialgebra $A(R)$, $\mathcal{D}$, called the quantum determinant and given by

$$\mathcal{D} = ad - bc + hac. \tag{4.2.11}$$

The commutation relations between $\mathcal{D}$ and the generators of $A(R)$ are

$$\begin{gathered}\mathcal{D}a = a\mathcal{D} + (h-g)c\mathcal{D}, \quad \mathcal{D}d = d\mathcal{D} - (h-g)c\mathcal{D},\\ \mathcal{D}c = c\mathcal{D}, \quad \mathcal{D}b = b\mathcal{D} + (h-g)(d\mathcal{D} - a\mathcal{D} - (h-g)c\mathcal{D}),\end{gathered} \tag{4.2.12}$$

so we can localise with respect to the Ore set $S = \{\mathcal{D}^\alpha : \alpha \in \mathbb{Z}_{\geq 1}\}$, and define $GL_{h,g}(2) = A(R)[\mathcal{D}^{-1}]$, having extra commutation relations

$$\begin{gathered}a\mathcal{D}^{-1} = \mathcal{D}^{-1}a + (h-g)\mathcal{D}^{-1}c, \quad d\mathcal{D}^{-1} = \mathcal{D}^{-1}d - (h-g)\mathcal{D}^{-1}c,\\ c\mathcal{D}^{-1} = \mathcal{D}^{-1}c, \quad b\mathcal{D}^{-1} = \mathcal{D}^{-1}b + (h-g)(\mathcal{D}^{-1}d - \mathcal{D}^{-1}a - (h-g)\mathcal{D}^{-1}c),\end{gathered} \tag{4.2.13}$$

with

$$\Delta(\mathcal{D}^{-1}) = \mathcal{D}^{-1} \otimes \mathcal{D}^{-1}, \quad \epsilon(\mathcal{D}^{-1}) = 1. \tag{4.2.14}$$

$GL_{h,g}(2)$ is a Hopf algebra with the antipode given by

$$\begin{gathered}S(a) = \mathcal{D}^{-1}(d + gc), \quad S(b) = \mathcal{D}^{-1}(gd - ga - b + g^2c),\\ S(c) = -\mathcal{D}^{-1}c, \quad S(d) = \mathcal{D}^{-1}(a - gc), \quad S(\mathcal{D}^{-1}) = \mathcal{D}.\end{gathered} \tag{4.2.15}$$

The Hopf algebra $GL_{h,g}(2)$ is clearly still polynomial with basis $\{\mathcal{D}^{-\alpha}b^\beta a^\gamma d^\delta c^\zeta : \alpha, \beta, \gamma, \delta, \zeta \in \mathbb{Z}_{\geq 0}\}$.

With $g = h$, $\mathcal{D}$ is central and we can consistently set $\mathcal{D} = 1$ and pass to the quantum group $SL_h(2)$. The relations for $SL_h(2)$ are just (4.2.5), but with the combination ad replaced wherever it appears by $bc - hac + 1$ and also the further relation $ad = bc - hac + 1$. With $g = h = 0$ we recover the classical group coordinate rings.

4.3. Review of Woronowicz's bicovariant differential calculus

We begin with the basic definitions.

DEFINITION 4.3.1. A *first order differential calculus* over an algebra A is a pair (Γ, d) such that:

1. Γ is an A-bimodule, i.e.
$$(a\omega)b = a(\omega b) \tag{4.3.1}$$
for all $a, b \in A$, $\omega \in \Gamma$, where the left and right actions which make Γ respectively a left A-module and right A-module are written multiplicatively.
2. d is a linear map, $\mathrm{d} : A \to \Gamma$.
3. For any $a, b \in A$, the Leibniz rule is satisfied, i.e.
$$\mathrm{d}(ab) = \mathrm{d}(a)b + a\mathrm{d}(b). \tag{4.3.2}$$
4. The bimodule Γ, or 'space of 1-forms', is spanned by elements of the form $a\mathrm{d}b$, $a, b \in A$.

REMARK 4.3.2. Given two first order differential calculi over an algebra A, (Γ, d) and (Γ', d'), we say that they are *isomorphic* if there is a bimodule isomorphism $\phi : \Gamma \to \Gamma'$ such that $\phi \circ \mathrm{d} = \mathrm{d}'$.

REMARK 4.3.3. We will usually write $\mathrm{d}a$ for $\mathrm{d}(a)$.

DEFINITION 4.3.4. A *bicovariant bimodule* over a Hopf algebra A is a triple $(\Gamma, \Delta_A^L, \Delta_A^R)$ such that:

1. Γ is an A-bimodule.
2. Γ is an A-bicomodule with left and right coactions Δ_A^L and Δ_A^R respectively, i.e.
$$(\mathrm{id} \otimes \Delta_A^L) \circ \Delta_A^L = (\Delta \otimes \mathrm{id}) \circ \Delta_A^L, \quad (\epsilon \otimes \mathrm{id}) \circ \Delta_A^L = \mathrm{id}, \tag{4.3.3}$$
making Γ a left A-comodule,
$$(\Delta_A^R \otimes \mathrm{id}) \circ \Delta_A^R = (\mathrm{id} \otimes \Delta) \circ \Delta_A^R, \quad (\mathrm{id} \otimes \epsilon) \circ \Delta_A^R = \mathrm{id}, \tag{4.3.4}$$
making Γ a right A-comodule, and
$$(\mathrm{id} \otimes \Delta_A^R) \circ \Delta_A^L = (\Delta_A^L \otimes \mathrm{id}) \circ \Delta_A^R, \tag{4.3.5}$$
which is the A-bicomodule property.
3. The coactions, Δ_A^L and Δ_A^R are bimodule maps, i.e.
$$\Delta_A^L(a\omega b) = \Delta(a)\Delta_A^L(\omega)\Delta(b), \tag{4.3.6}$$
$$\Delta_A^R(a\omega b) = \Delta(a)\Delta_A^R(\omega)\Delta(b). \tag{4.3.7}$$

REMARK 4.3.5. The Sweedler notation for coproducts in the Hopf algebra A is taken to be $\Delta(a) = a_{(1)} \otimes a_{(2)}$ for all $a \in A$ and is extended to the coactions as $\Delta_A^L(\omega) = \omega_{(A)} \otimes \omega_{(\Gamma)}$ and $\Delta_A^R(\omega) = \omega_{(\Gamma)} \otimes \omega_{(A)}$. In this notation the conditions (4.3.3), (4.3.4) and (4.3.5) become
$$\omega_{(A)} \otimes (\omega_{(\Gamma)})_{(A)} \otimes (\omega_{(\Gamma)})_{(\Gamma)} = (\omega_{(A)})_{(1)} \otimes (\omega_{(A)})_{(2)} \otimes \omega_{(\Gamma)}, \quad \epsilon(\omega_{(A)})\omega_{(\Gamma)} = \omega, \tag{4.3.8}$$
$$(\omega_{(\Gamma)})_{(\Gamma)} \otimes (\omega_{(\Gamma)})_{(A)} \otimes \omega_{(A)} = \omega_{(\Gamma)} \otimes (\omega_{(A)})_{(1)} \otimes (\omega_{(A)})_{(2)}, \quad \epsilon(\omega_{(A)})\omega_{(\Gamma)} = \omega, \tag{4.3.9}$$
$$\omega_{(A)} \otimes (\omega_{(\Gamma)})_{(\Gamma)} \otimes (\omega_{(\Gamma)})_{(A)} = (\omega_{(\Gamma)})_{(A)} \otimes (\omega_{(\Gamma)})_{(\Gamma)} \otimes \omega_{(A)}, \tag{4.3.10}$$
for all $\omega \in \Gamma$.

DEFINITION 4.3.6. A *first order bicovariant differential calculus* over a Hopf algebra A is a quadruple $(\Gamma, \mathrm{d}, \Delta_A^L, \Delta_A^R)$ such that:

1. (Γ, d) is a first order differential calculus over A.
2. $(\Gamma, \Delta_A^L, \Delta_A^R)$ is a bicovariant bimodule over A.

3. d is both a left and a right comodule map, i.e.

$$(\mathrm{id} \otimes \mathrm{d}) \circ \Delta(a) = \Delta_A^L(\mathrm{d}a), \tag{4.3.11}$$

$$(\mathrm{d} \otimes \mathrm{id}) \circ \Delta(a) = \Delta_A^R(\mathrm{d}a), \tag{4.3.12}$$

for all $a \in A$.

REMARK 4.3.7. Given a first order calculus over a Hopf algebra, (Γ, d), (4.3.11) and (4.3.12) uniquely determine left and right coactions, and hence a bicovariant bimodule structure. However the *existence* of these coactions and the corresponding bicovariant bimodule is of course *not* guaranteed.

EXAMPLE 4.3.8. Given any associative algebra A, we may form a first order differential calculus over A, (A^2, D), where

$$A^2 = \left\{ \sum_k a_k \otimes b_k \in A \otimes A : \sum_k a_k b_k = 0 \right\}, \tag{4.3.13}$$

and $\mathrm{D} : A \to A^2$ is given by

$$\mathrm{D}a = 1 \otimes a - a \otimes 1. \tag{4.3.14}$$

A^2 has a bimodule structure given, for all $a_k, b_k, c \in A$ by

$$c\left(\sum_k a_k \otimes b_k\right) = \sum_k c a_k \otimes b_k, \quad \left(\sum_k a_k \otimes b_k\right) c = \sum_k a_k \otimes b_k c. \tag{4.3.15}$$

The importance of this differential calculus lies in the fact that *any* first order differential calculus over A, (Γ, d), is isomorphic to one of the form $(A^2/\mathcal{N}, \pi \circ \mathrm{D})$ where $\mathcal{N}$ is the kernel of the surjective map $\pi : A^2 \to \Gamma$ defined by $\pi(\sum_k a_k \otimes b_k) = \sum a_k \mathrm{d} b_k$. For this reason, (A^2, D) is said to be *universal*. Moreover, it is not difficult to check that when A is a Hopf algebra, (A^2, D) is a first order bicovariant differential calculus.

DEFINITION 4.3.9. An element ω of a bicovariant bimodule $(\Gamma, \Delta_A^L, \Delta_A^R)$ is said to be *left-invariant* if

$$\Delta_A^L(\omega) = 1 \otimes \omega, \tag{4.3.16}$$

right-invariant if

$$\Delta_A^R(\omega) = \omega \otimes 1, \tag{4.3.17}$$

and *bi-invariant* if it is both left- and right-invariant.

REMARK 4.3.10. Denoting the vector space of all left-invariant elements of Γ by Γ_{inv}, there is a projection $P : \Gamma \to \Gamma_{\mathrm{inv}}$ defined on any element $\omega \in \Gamma$ as,

$$P(\omega) = S(\omega_{(A)})\omega_{(\Gamma)}. \tag{4.3.18}$$

If $(\Gamma, \mathrm{d}, \Delta_A^L, \Delta_A^R)$ is a first order bicovariant differential calculus over A then it is not difficult to see the equivalence of the statements that the differentials generate Γ as a left A-module and that the elements $P(\mathrm{d}a) = S(a_{(1)})\mathrm{d}a_{(2)}$, for all $a \in A$, span the vector space of left-invariant 1-forms. Further, for any $a \in A$, $\mathrm{d}a$ may be expressed in terms of left-invariant forms as

$$\mathrm{d}a = a_{(1)} P(\mathrm{d}a_{(2)}). \tag{4.3.19}$$

An alternative construction of the universal differential calculus occurs in the situation of particular interest to us. It is described in the following example.

EXAMPLE 4.3.11. We start with a Hopf algebra A defined in terms of a finite number of generators, $(T_{ij})_{i,j=1\dots n}$ say, together with certain relations which are consistent with a PBW-type basis and the usual (matrix-element bialgebra) coproduct and counit. We may then introduce a bimodule Γ_0 as the free A-bimodule on the symbols d_0T_{ij} with the linear map $\mathrm{d}_0 : A \to \Gamma$ defined on any element of A by way of the Leibniz rule. (Γ_0, d_0) is then also a universal first order differential calculus, and being therefore isomorphic to (A^2, D) is certainly bicovariant. Indeed, (4.3.11) and (4.3.12) specify the coactions, and the space of left invariant forms, $\Gamma_{0\mathrm{inv}}$, is spanned by all elements of the form $S(a_{(1)})\theta_{ik}a_{(2)}$ where $\theta_{ik} = \sum_j S(T_{ij})\mathrm{d}_0T_{jk}$ and $a \in A$. If we denote by $(\vartheta_i)_{i\in I}$ a basis for Γ_0, where I is some countably infinite index set, then from (4.3.12), there exist v_{ij}s such that the right coaction on the ϑ_is takes the form

$$\Delta^R_A(\vartheta_i) = \sum_{j\in I} \vartheta_j \otimes v_{ji}. \tag{4.3.20}$$

In particular, the n^2 left invariant elements θ_{ik} span a sub-bicomodule, with

$$\Delta^R_A(\theta_{ik}) = \sum_{s,t=1\dots n} \theta_{st} \otimes S(T_{is})T_{tk}. \tag{4.3.21}$$

At this point in Example 4.3.11 the only relations between algebra and bimodule elements are those coming from the Leibniz rule. Obtaining further relations involves finding a suitable 'relation space', $\mathcal{N}$, which can be factored out from Γ_0 while maintaining the bicovariance. In the context of the universal calculus, (A^2, D), Theorems 1.5 and 1.8 of **[104]** tell us that such $\mathcal{N}$ must be of the form $\tau^{-1}(A\otimes\mathcal{R})$ where $\tau^{-1}(a\otimes b) = aS(b_{(1)})\otimes b_{(2)}$ and $\mathcal{R}$ is a right ideal of A, contained in $\ker\epsilon$, which is stable under the right adjoint coaction[1]. Conversely, given a first order bicovariant differential calculus, $(\Gamma, \mathrm{d}, \Delta^L_A, \Delta^R_A)$, $\mathcal{R}$ may be recovered as the set of all $a \in \ker\epsilon$ such that $P(\mathrm{d}a) = 0$.

It is desirable to classify all bicovariant calculi on a given quantum group which have particular properties. This problem of classification can be regarded as the problem of classifying the ad-invariant ideals $\mathcal{R}$ **[88, 89, 49]**. However, following Müller-Hoissen **[77]**, we will consider a more 'hands on' approach. We look for calculi whose bimodule of 1-forms is generated as a *left* A-module by the differentials of the generators (this assumption is also made in **[88, 89]**). In effect we are passing directly to a class of, as yet unspecified, quotients of Γ_0 which we then wish to constrain by the requirement that the bicovariance is not destroyed. For this approach we need the characterisation of bicovariant bimodules which is provided by the following theorem of Woronowicz.

THEOREM 4.3.12. *Let $(\Gamma, \Delta^L_A, \Delta^R_A)$ be a bicovariant bimodule over A and let $(\theta_i)_{i\in I}$ be a basis of Γ_{inv}, where I is some countable index set. Then we have:*

1. Any element $\omega \in \Gamma$ has a unique expression as

$$\omega = \sum_{i\in I} a_i\theta_i, \tag{4.3.22}$$

where $a_i \in A$.

[1] We recall that the right adjoint coaction is defined on any $a \in A$ as $Ad^*_R(a) = a_{(2)} \otimes S(a_{(1)})a_{(3)}$

2. *There exist linear functionals $f_{ij} \in A^*$, $i, j \in I$, such that*

$$\theta_i a = \sum_{j \in I} (f_{ij} \star a)\theta_j, \tag{4.3.23}$$

where $\alpha \star a = \alpha(a_{(2)})a_{(1)}$ for all $\alpha \in A^*$, $a \in A$.

3. *The functionals f_{ij} are uniquely determined by (4.3.23) and satisfy*

$$f_{ik}(ab) = \sum_{j \in I} f_{ij}(a) f_{jk}(b), \quad f_{ik}(1) = \delta_{ik}, \tag{4.3.24}$$

for all $a, b \in A$, $i, j \in I$.

4. *There exist elements $v_{ij} \in A$, $i, j \in I$, such that for all t_i*

$$\Delta_A^R(\theta_i) = \sum_{j \in I} \theta_j \otimes v_{ji}, \tag{4.3.25}$$

$$\Delta(v_{ik}) = \sum_{j \in I} v_{ij} \otimes v_{jk}, \quad \epsilon(v_{ik}) = \delta_{ik}. \tag{4.3.26}$$

5. *For all $a \in A$*

$$\sum_{j \in I} v_{ji}(a \star f_{jk}) = \sum_{j \in I} (f_{ij} \star a) v_{kj}, \tag{4.3.27}$$

where $a \star \alpha = \alpha(a_{(1)})a_{(2)}$ for all $\alpha \in A^*$, $a \in A$.

Conversely, if $(\theta_i)_{i \in I}$ is a basis of a vector space V, and we have functionals $(f_{ij})_{i,j \in I}$ defined on A and elements $(v_{ij})_{i,j \in I}$ in A which satisfy (4.3.24), (4.3.26) and (4.3.27), then there exists a unique bicovariant bimodule such that $V = \Gamma_{inv}$ and (4.3.23) and (4.3.25) are satisfied.

This result places significant constraints on the possible bicovariant calculi which are consistent with our assumption that the differentials of the generators should generate the bimodule of forms as a left A-module. That assumption implies immediately that Γ_{inv} is spanned by the *finite* set of elements θ_{ik}. Choosing a basis from this set we either take all n^2 as linear independent elements, in which case the v_{ij}s of (4.3.25) and (4.3.26) have already been determined in (4.3.21), or we introduce linear relations between the θ_{ik}s. In the latter case we must be sure that such relations do not destroy the bicovariance — they must factor through the coactions. This condition will tend to fix the possible bases of Γ_{inv}, which in turn determines the v_{ij}s through (4.3.25) and (4.3.21). It then follows immediately that (4.3.26) is satisfied. Now, trying to introduce commutation relations which factor through the left and right coactions which we wish to maintain, it is not too difficult to see that when these relations take the form (4.3.23) with the f_{ij}s satisfying (4.3.24) and (4.3.27), bicovariance is maintained. Moreover, as the theorem states, the commutation relations in *any* bicovariant bimodule must *always* take this form. Thus our method of attacking the classification problem can now be discerned. We decide upon a valid basis of Γ_{inv}, and then assume the general form for the commutation relations, (4.3.23). We must then impose as constraints, consistency with the relations already present from the Leibniz rule, together with (4.3.24) and (4.3.27). This procedure will be made absolutely explicit for the case of the quantum groups of particular interest to us in the following section.

REMARK 4.3.13. Having chosen a basis $(\theta_i)_{i\in I}$ for $\Gamma_{\rm inv}$, the right-invariant elements $(\eta_i)_{i\in I}$ defined by $\eta_i = \sum_{j\in I}\theta_j S(v_{ji})$ form a basis for the vector space of right invariant 1-forms, $\Gamma_{\rm rinv}$.

REMARK 4.3.14. The *dimension* of a bicovariant calculus is defined to be $\dim \Gamma_{\rm inv}$. We will only be interested in finite, d-dimensional, examples so we will eschew the index set and consider indices running over a finite set.

Given two bicovariant bimodules, $(\Gamma, \Delta^L_A, \Delta^R_A)$ and $(\tilde{\Gamma}, \tilde{\Delta}^L_A, \tilde{\Delta}^R_A)$, of dimensions d and $\tilde{d}$ respectively, their tensor product over A, $(\Gamma' = \Gamma\otimes_A\tilde{\Gamma}, \Delta'^L_A, \Delta'^R_A)$ is also a bicovariant bimodule as follows. The left and right coactions of A on Γ' are given by

$$\Delta'^L_A(\omega\otimes\tilde{\omega}) = \omega_{(A)}\tilde{\omega}_{(A)}\otimes\omega_{(\Gamma)}\otimes\tilde{\omega}_{(\Gamma)}, \quad \Delta'^R_A(\omega\otimes\tilde{\omega}) = \omega_{(\Gamma)}\otimes\tilde{\omega}_{(\Gamma)}\otimes\omega_{(A)}\tilde{\omega}_{(A)}, \tag{4.3.28}$$

and the f functionals of Theorem 4.3.12 are now given by $F'_{ik,jl} = f_{ij} * \tilde{f}_{kl}$, where $*$ is the usual convolution product, such that

$$(\theta_i\otimes\tilde{\theta}_k)a = \sum_{j=1\ldots d, l=1\ldots\tilde{d}} (F'_{ik,jl}\star a)(\theta_j\otimes\tilde{\theta}_l), \tag{4.3.29}$$

for all $a\in A$, where $(\theta_i)_{i=1\ldots d}$ and $(\tilde{\theta}_i)_{i=1\ldots\tilde{d}}$ are the bases of left invariant elements in $\Gamma_{\rm inv}$ and $\tilde{\Gamma}_{\rm inv}$ respectively, so that $(\theta_i\otimes\tilde{\theta}_j)_{i=1\ldots d, j=1\ldots\tilde{d}}$ is the basis of left invariant elements in $\Gamma'_{\rm inv}$. It is clear that in this way we can build arbitrary n-fold tensor powers, $\Gamma^{\otimes n} = \Gamma\otimes_A\Gamma\otimes_A\ldots\otimes_A\Gamma$ of a given bicovariant bimodule, all of which are themselves bicovariant with left and right coactions denoted by Δ^{nL}_A and Δ^{nR}_A respectively. We can then define $\Gamma^{\otimes} = A\oplus\Gamma\oplus\Gamma^{\otimes 2}\oplus\ldots$ to be the analog of the classical algebra of covariant tensor fields. $\Gamma^{\otimes}$ is a bicovariant graded algebra, in that it is a tensor algebra and a bicovariant bimodule over A, with coactions $\Delta^{\otimes L}_A$ and $\Delta^{\otimes R}_A$ which are algebra maps and coincide on elements of $\Gamma^{\otimes n}$ with the coactions Δ^{nL}_A and Δ^{nR}_A respectively.

The next step in Woronowicz's construction of a noncommutative geometry for Hopf algebras is to introduce an analogue of the classical external algebra of forms. Starting from the bicovariant graded algebra $\Gamma^{\otimes}$ we want to obtain another bicovariant graded algebra, the *external bicovariant graded algebra*, Ω, as a quotient, $\Omega = \Gamma^{\otimes}/S$, by some graded two-sided ideal S. Woronowicz introduces the *unique* linear bimodule map $\Lambda : \Gamma\otimes\Gamma\to\Gamma\otimes\Gamma$ such that

$$\Lambda(\theta\otimes\eta) = \eta\otimes\theta \tag{4.3.30}$$

for any $\theta\in\Gamma_{\rm inv}$, $\eta\in\Gamma_{\rm rinv}$. Λ is then also a bicomodule map, is given explicitly on the basis of left invariant elements $(\theta_i\otimes\theta_k)_{i,j=1\ldots d}$ by

$$\Lambda(\theta_i\otimes\theta_k) = \sum_{s,t=1\ldots d}\Lambda_{ik,st}\theta_s\otimes\theta_t = \sum_{s,t=1\ldots d} f_{it}(v_{sk})\theta_s\otimes\theta_t, \tag{4.3.31}$$

and can be shown to satisfy the braid equation, $\Lambda_{12}\Lambda_{23}\Lambda_{12} = \Lambda_{23}\Lambda_{12}\Lambda_{23}$. The map Λ may then be used in just the same way as the permutation operator is used classically. Thus we define an analogue of the antisymmetrisation operator on $\Gamma^{\otimes n}$, $W_{1\ldots n}$, by replacing the classical permutation operator by Λ everywhere in the classical antisymmetriser. $W_{1\ldots n}$ is then a bimodule and bicomodule map $W_{1\ldots n} : \Gamma^{\otimes n}\to\Gamma^{\otimes n}$ and we can define $S^n = \ker W_{1\ldots n}$ so that $\Gamma^{\otimes n}/S^n$ is isomorphic to $\operatorname{Im} W_{1\ldots n}$ and $\Omega = \Gamma^{\otimes}/S$ where $S = \bigoplus_{n=2} S^n$. As $W_{1\ldots n}$ is a bicomodule map, the left and right coactions of $\Gamma^{\otimes}$ descend to the quotient where they shall be denoted $\Delta^{\Omega L}_A$ and $\Delta^{\Omega R}_A$ respectively. Moreover, we can now define the *wedge product* as $\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_n = W_{1\ldots n}(\omega_1 \otimes \omega_2 \otimes \ldots \otimes \omega_n)$ where the ω_i are arbitrary elements of Γ.

REMARK 4.3.15. If we were to construct the bicovariant graded algebra $\tilde{\Gamma}^{\otimes}$ from a bicovariant bimodule $\tilde{\Gamma}$ which contains Γ as a sub-bimodule and a sub-bicomodule , then the bicovariant graded algebra $\Gamma^{\otimes}$ embeds naturally into $\tilde{\Gamma}^{\otimes}$ in the sense that there is the obvious natural embedding map $\phi : \Gamma^{\otimes} \to \tilde{\Gamma}^{\otimes}$ such that

$$\phi|_A = \mathrm{id}, \quad \phi|_\Gamma = \iota, \tag{4.3.32}$$

$$\phi(\omega\rho) = \phi(\omega)\phi(\rho), \tag{4.3.33}$$

$$(\mathrm{id} \otimes \phi) \circ \Delta^L_A = \Delta^{\otimes L}{}_A \circ \phi, \quad (\phi \otimes \mathrm{id}) \circ \Delta^R_A = \Delta^{\otimes R}{}_A \circ \phi, \tag{4.3.34}$$

where $\iota : \Gamma \to \tilde{\Gamma}$ is the natural inclusion. An attractive feature of the external bicovariant algebra construction, which has its origin in the uniqueness of the map Λ, is that this embedding survives the quotienting procedures so that $(\Omega, \Delta^{\Omega L}{}_A, \Delta^{\Omega R}{}_A)$ embeds naturally in $(\tilde{\Omega}, \Delta^{\tilde{\Omega} L}{}_A, \Delta^{\tilde{\Omega} R}{}_A)$.

We may now state the following theorem of Woronowicz:

THEOREM 4.3.16. *Let Ω be the external bicovariant algebra constructed above. There exists one and only one linear map* $\mathrm{d} : \Omega \to \Omega$ *such that:*

1. d *increases the grade by one.*
2. *On elements of grade 0,* d *coincides with the differential of the first order bicovariant calculus* $(\Gamma, \mathrm{d}, \Delta^L_A, \Delta^R_A)$.
3. *For all $\omega \in \Gamma^{\otimes k}$, $k = 0, 1, 2, \ldots,$ and $\omega' \in \Omega$*
$$\mathrm{d}(\omega \wedge \omega') = \mathrm{d}\omega \wedge \omega' + (-1)^k \omega \wedge \mathrm{d}\omega'. \tag{4.3.35}$$
4. *For any $\omega \in \Omega$,*
$$\mathrm{d}(\mathrm{d}\omega) = 0. \tag{4.3.36}$$
5. d *is a both a left and a right comodule map i.e.*
$$\Delta^{\Omega L}{}_A(\mathrm{d}\omega) = (\mathrm{id} \otimes \mathrm{d}) \circ \Delta^{\Omega L}{}_A(\omega), \tag{4.3.37}$$
$$\Delta^{\Omega R}{}_A(\mathrm{d}\omega) = (\mathrm{d} \otimes \mathrm{id}) \circ \Delta^{\Omega R}{}_A(\omega), \tag{4.3.38}$$

for all $\omega \in \Omega$.

REMARK 4.3.17. The external bicovariant algebra Ω equipped with the differential described in this result will be called the *exterior bicovariant differential calculus* over the Hopf algebra A, and denoted by $(\Omega, \mathrm{d}, \Delta^{\Omega L}{}_A, \Delta^{\Omega R}{}_A)$. Brzeziński has shown that this is a super-Hopf algebra [**15**].

REMARK 4.3.18. To prove this theorem Woronowicz extends the bimodule Γ to $\tilde{\Gamma} = Ax \oplus \Gamma$ where Ax is the left A-module freely generated by the single element x, such that the right action of A on an element ax is given by $axb = abx + a\mathrm{d}b$ and the coactions are such that the element x is bi-invariant. The theorem is then proved for the external bicovariant algebra $\tilde{\Omega}$ built from $\tilde{\Gamma}$ with the final result for Ω coming after using the natural embedding of Ω in $\tilde{\Omega}$. Along the way the differential is expressed as $\mathrm{d}a = [x, a]$, for any $a \in A$, and more generally as $\mathrm{d}\omega = [x, \omega]_\mp$ for any $\omega \in \tilde{\Gamma}$ where $[x, \omega]_\mp = x \wedge \omega \mp \omega \wedge x$ with $-$ and $+$ for ω of even and odd grade respectively. In particular examples of bicovariant differential calculi it sometimes happens that there is a bi-invariant element *within* the unextended calculus which implements the differential in this way. Such calculi are called *inner*.

The final component of the Woronowicz differential calculus is the analogue of the classical Lie algebra of tangent vectors at the identity. Classically this is isomorphic to $(\ker\epsilon/(\ker\epsilon)^2)^*$. In the abstract Hopf algebra setting it turns out that defining a space $\mathcal{L}$ by $\mathcal{L} = (\ker\epsilon/\mathcal{R})^*$ there is a unique bilinear map, $\langle,\rangle : \Gamma \times \mathcal{L} \to \mathbb{C}$, such that for all $a \in A$, $\omega \in \Gamma$ and $\chi \in \mathcal{L}$

$$\langle a\omega, \chi\rangle = \epsilon(a)\langle\omega,\chi\rangle, \quad \langle \mathrm{d}a, \chi\rangle = \chi(a), \tag{4.3.39}$$

which is *non-degenerate* as a pairing between Γ_{inv} and T. Thus, we can introduce a basis $(\chi_i)_{i=1\dots d}$ for $\mathcal{L}$ dual to $(\theta_i)_{i=1\dots d}$. Also, it is not too difficult to see that for any $\omega \in \Gamma$ there is an element $a \in \ker\epsilon$ such that $P(\omega) = P(\mathrm{d}a)$. Combined with the defining characteristics of the pairing between Γ and $\mathcal{L}$, we then arrive at a basis a_i of $(\ker\epsilon/\mathcal{R})$ such that $\theta_i = P(\mathrm{d}a_i)$. These a_i are analogues of the classical coordinate functions at the identity. The following result of Woronowicz continues the analogy with the classical situation.

THEOREM 4.3.19. *For all $a, b \in A$*

$$\mathrm{d}a = \sum_{j=1\dots d} (\chi_j \star a)\theta_j, \tag{4.3.40}$$

$$\mathrm{d}\theta_i = -\sum_{j,l=1\dots d} \mathcal{C}_{jl,i}\theta_j \wedge \theta_l, \tag{4.3.41}$$

$$\chi_i(ab) = \sum_{j=1\dots d} \chi_j(a) f_{ji}(b) + \epsilon(a)\chi_i(b), \tag{4.3.42}$$

$$\sum_{j=1\dots d} \chi_j(a)v_{ij} = \chi_i(a_{(2)})S(a_{(1)})a_{(3)}, \tag{4.3.43}$$

where the f_{ji} and v_{ij} are as introduced in Theorem 4.3.12, and $\mathcal{C}_{jl,i} = (\chi_j * \chi_l)(a_i)$.

The equation (4.3.42) shows us that the χ_i may be interpreted as 'deformed derivations' and is equivalent to a coproduct for the 'quantum tangent vectors'

$$\Delta(\chi_i) = \sum_{j=1\dots d} \chi_j \otimes f_{ji} + 1 \otimes \chi_i. \tag{4.3.44}$$

Further, as $\mathcal{R}$ is stable under the right adjoint coaction, it follows that $\mathcal{L}$ is stable under the right adjoint A^*-action, $\overset{\text{ad}}{\triangleleft}$, that is, $S(\alpha_{(1)})\chi\alpha_{(2)} \in \mathcal{L}$ for all $\alpha \in A^*$ and any $\chi \in \mathcal{L}$. In the classical case this action, restricted to the tangent space, provides the Lie bracket. So we may define a *quantum Lie bracket* as

$$[\chi_i, \chi_k] = \chi_i \overset{\text{ad}}{\triangleleft} \chi_k = \sum_{j=1\dots d} C_{ik,j}\chi_j, \tag{4.3.45}$$

where the $C_{ik,j}$ are analogues of the classical Lie algebra structure constants and are still to be determined. But from the coproduct on the χ_i we can expand the right adjoint A^*-action, to obtain

$$[\chi_i, \chi_k] = \chi_i\chi_k - \sum_{s=1\dots d} \chi_s(\chi_i \overset{\text{ad}}{\triangleleft} f_{sk}). \tag{4.3.46}$$

We can then use (4.3.43) to determine $\chi_i \overset{\mathrm{ad}}{\triangleleft} f_{sk}$ and obtain an expression for the bracket as a *quantum commutator*

$$[\chi_i, \chi_k] = \chi_i\chi_k - \sum_{s,t=1\ldots d} \Lambda_{st,ik}\chi_s\chi_t. \tag{4.3.47}$$

The structure constants, $\mathcal{C}_{ik,j}$, may now be determined by evaluating the right hand sides of (4.3.45) and (4.3.47) on a_l and equating the results to obtain

$$\mathcal{C}_{ik,j} = \mathfrak{C}_{ik,j} - \sum_{s,t=1\ldots d} \Lambda_{st,ik}\mathfrak{C}_{st,j}. \tag{4.3.48}$$

There is a *quantum Jacobi identity* for the quantum Lie bracket given by

$$[\chi_i, [\chi_j, \chi_k]] = [[\chi_i, \chi_j], \chi_k] - \sum_{s,t=1\ldots d} \Lambda_{st,jk}[[\chi_i, \chi_s], \chi_t]. \tag{4.3.49}$$

The *universal enveloping algebra* of the quantum Lie algebra $\mathcal{L}$ may be introduced as the quotient of the tensor algebra of $\mathcal{L}$ by the two-sided ideal generated by the elements $\chi_i\chi_k - \sum_{s,t=1\ldots d}\Lambda_{st,ik}\chi_s\chi_t - \sum_{j=1\ldots d}\mathcal{C}_{ik,j}\chi_j$.

4.4. The classification procedure

In this section we make explicit the procedure, already outlined in the previous section, for determining under certain assumptions all possible first order bicovariant differential calculi for a given quantum group. It was first applied by Müller-Hoissen [**77, 78**] to the case of the standard 2-parameter quantum group $GL_{q,p}(2)$. Some further results appeared in subsequent papers [**79, 80**], and in [**13**] it was applied to the standard 1-parameter quantum group $GL_q(3)$. Here we apply this 'recipe' to the cases of $GL_{h,g}(2)$ and $SL_h(2)$.

Starting with our quantum group A, where A is either $GL_{h,g}(2)$ or $SL_h(2)$, we introduce the first order differential calculus, in the first instance, as the free A-bimodule Γ_0, on the symbols $\{\mathrm{d}_0a, \mathrm{d}_0b, \mathrm{d}_0c, \mathrm{d}_0d\}$ with the differential $\mathrm{d}_0 : GL_{h,g}(2) \to \Gamma_0$ defined on any element of A by way of the Leibniz rule. Note that $\mathrm{d}_0\mathcal{D}^{-1} = -\mathcal{D}^{-1}\mathrm{d}\mathcal{D}\mathcal{D}^{-1}$. However, as already mentioned, we pass directly to some quotient (Γ, d) which we assume is generated as a left A-module by $\{\mathrm{d}a, \mathrm{d}b, \mathrm{d}c, \mathrm{d}d\}$ and is still a bicovariant bimodule, denoted $(\Gamma, \mathrm{d}, \Delta_A^L, \Delta_A^R)$. Then, by Remark 4.3.10 Γ_{inv} is spanned by the 4 left-invariant forms $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ where

$$\begin{aligned} \theta_1 = P(\mathrm{d}a) = S(a)\mathrm{d}a + S(b)\mathrm{d}c, \quad \theta_2 = P(\mathrm{d}b) = S(a)\mathrm{d}b + S(b)\mathrm{d}d,\\ \theta_3 = P(\mathrm{d}c) = S(c)\mathrm{d}a + S(d)\mathrm{d}c, \quad \theta_4 = P(\mathrm{d}d) = S(c)\mathrm{d}b + S(d)\mathrm{d}d. \end{aligned} \tag{4.4.1}$$

In the other direction, from (4.3.19), the differentials of the generators may be written in terms of the left invariant θ_is as,

$$\begin{aligned} \mathrm{d}a = a_{(1)}P(\mathrm{d}a_{(2)}) = a\theta_1 + b\theta_3, \quad \mathrm{d}b = b_{(1)}P(\mathrm{d}b_{(2)}) = a\theta_2 + b\theta_4,\\ \mathrm{d}c = c_{(1)}P(\mathrm{d}c_{(2)}) = c\theta_1 + d\theta_3, \quad \mathrm{d}d = d_{(1)}P(\mathrm{d}d_{(2)}) = c\theta_2 + d\theta_4. \end{aligned} \tag{4.4.2}$$

We are now free to choose a basis for Γ_{inv} from $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ and look for corresponding non-trivial first order bicovariant differential calculi. We choose to look for calculi with the classical dimension, so for $GL_{h,g}(2)$ we will make the further assumption that $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ is a basis of Γ_{inv} and call any extant calculi 4D calculi, while for $SL_h(2)$ we will look for 3-dimensional, 3D, calculi by assuming that $\{\theta_1, \theta_2, \theta_3\}$ is a basis of G_{inv}

with $\theta_4 = \alpha_1\theta_1 + \alpha_2\theta_2 + \alpha_3\theta_3$ and the coefficients α_1, α_2 and α_3 to be determined. The assumptions will be justified if we find non-trivial calculi. This in turn will be established if we can find functionals f_{ij} and elements v_{ij} as in Theorem 4.3.12 consistent with our assumptions.

As we discussed in the previous section, the elements v_{ij} are already fixed through our assumptions, by (4.3.12), (4.4.1) and (4.3.25). For the 4D, $GL_{h,g}(2)$, calculi, we have

$$V = ||v_{ij}|| = \begin{pmatrix} S(a)a & S(a)b & S(c)a & S(c)b \\ S(a)c & S(a)d & S(c)c & S(c)d \\ S(b)a & S(b)b & S(d)a & S(d)b \\ S(b)c & S(b)d & S(d)c & S(d)d \end{pmatrix}. \tag{4.4.3}$$

For the 3D, $SL_h(2)$, calculi we have

$$V = ||v_{ij}|| = \begin{pmatrix} S(a)a + \alpha_1 S(b)c & S(a)b + \alpha_1 S(b)d & S(c)a + \alpha_1 S(d)c \\ S(a)c + \alpha_2 S(b)c & S(a)d + \alpha_2 S(b)d & S(c)c + \alpha_2 S(d)c \\ S(b)a + \alpha_3 S(b)c & S(b)b + \alpha_3 S(b)d & S(d)a + \alpha_3 S(d)c \end{pmatrix}, \tag{4.4.4}$$

but in this case we must also have

$$\Delta_A^R(\theta_4 - \alpha_1\theta_1 + \alpha_2\theta_2 + \alpha_3\theta_3) = 0. \tag{4.4.5}$$

Evaluating (4.4.5) with (4.4.1), (4.3.12) and (4.4.2) fixes the α_i coefficients to be given by $\alpha_1 = -1$, $\alpha_2 = 0$ and $\alpha_3 = -2h$, so that we must have

$$\theta_4 = -\theta_1 - 2h\theta_3. \tag{4.4.6}$$

In fact, when we look for bi-invariant forms in the 4D calculi, we soon find that up to scalar multiplication there is but one, which we will denote by $\mathrm{Tr}_h\,\Theta$, and which is given by

$$\mathrm{Tr}_h\,\Theta = \theta_1 + 2h\theta_3 + \theta_4 \tag{4.4.7}$$

Existence of the bicovariant first order differential calculi which we seek now hinges entirely on the f_{ij}s. Following Müller-Hoissen [**77**], let us refine our notation slightly. For the 4D calculi, we write the relations (4.3.23) in terms of four 4×4 matrices, the '$ABCD$' matrices, $A_{ij} = f_{ij}(a)$, $B_{ij} = f_{ij}(b)$, $C_{ij} = f_{ij}(c)$ and $D_{ij} = f_{ij}(d)$, as

$$\begin{aligned} \theta_i a &= \sum_{j=1\ldots4} (aA_{ij} + bC_{ij})\theta_j, \quad \theta_i b = \sum_{j=1\ldots4} (aB_{ij} + bD_{ij})\theta_j, \\ \theta_i c &= \sum_{j=1\ldots4} (cA_{ij} + dC_{ij})t_j, \quad \theta_i d = \sum_{j=1\ldots4} (cB_{ij} + dD_{ij})\theta_j. \end{aligned} \tag{4.4.8}$$

In the case of the 3D calculi the 4×4 $ABCD$ matrices are simply replaced by 3×3 $ABCD$ matrices with the summations then to 3. We may now list the constraints on the $ABCD$ matrices.

Constraint 1: Differentiating the quantum group relations, (4.2.5), to obtain in R-matrix form

$$\mathrm{d}(RT_1T_2 - T_2T_1R) = R(\mathrm{d}T_1)T_2 + RT_1(\mathrm{d}T_2) - (\mathrm{d}T_2)T_1R - T_2(\mathrm{d}T_1)R = 0, \tag{4.4.9}$$

we replace the differentials by left-invariant forms through (4.4.2). We then use (4.4.8) to commute the $\theta_i s$ to the right which allows us then to equate the (ordered) algebra valued coefficients and obtain linear relations between the matrix elements of the $ABCD$ matrices. Similarly for the 3D case, but then, when replacing the differentials by left-invariant forms, we also use (4.4.6).

Constraint 2: In both the 3D and 4D cases the relations (4.3.27) may be expressed in the following matrix form

$$\begin{pmatrix} V^TA & V^TB \\ V^TC & V^TD \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} AV^T & BV^T \\ CV^T & DV^T \end{pmatrix}. \tag{4.4.10}$$

Only algebra elements appear in these equations and once all terms are 'straightened' they yield further linear relations among the matrix elements of the $ABCD$ matrices.

Constraint 3: The equations (4.3.24) are telling us that A, B, C and D must be the representation matrices of a, b, c and d respectively, and that the matrix representation of the determinant, $\mathbb{D}$ say, where $\mathbb{D} = AD - BC + hAC$, must be invertible in the case of $GL_{h,g}(2)$, and equal to the identity in the case of $SL_h(2)$. Imposing these conditions we obtain non-linear relations amongst the $ABCD$ matrix elements.

REMARK 4.4.1. Recall from Remark 4.3.2 our definition of isomorphism for differential calculi. As we assume a single θ_i basis, different possible $ABCD$ matrices must correspond to non-isomorphic calculi.

The $ABCD$ matrices which result from this procedure provide the most general possible first order bicovariant calculi under the stated assumptions. We may now investigate the external bicovariant graded algebras, and also the 'quantum Lie algebras' which are related to our first order calculi.

We begin by using Theorem 4.3.16, in particular (4.3.36), together with (4.4.2) to deduce the structure constants $\mathcal{C}_{ij,k}$ appearing in the 'Cartan-Maurer equations', (4.3.41). For the 4D calculi, we obtain four 4×4 matrices

$$\mathcal{C}_{ij,1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{C}_{ij,2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\mathcal{C}_{ij,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \mathcal{C}_{ij,4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.4.11}$$

while for the 3D calculi we obtain three 3×3 matrices

$$\mathcal{C}_{ij,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{C}_{ij,2} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -2h \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{C}_{ij,3} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -2h \end{pmatrix}, \tag{4.4.12}$$

together with a relation,

$$2\theta_1 \wedge \theta_1 + 4h\theta_3 \wedge \theta_1 + \theta_2 \wedge \theta_3 + \theta_3 \wedge \theta_2 = 0, \tag{4.4.13}$$

which will have to be consistent with any commutation relations which we derive for left-invariant forms of the 3D calculi. In fact these commutation relations can now be obtained by differentiating the relations (4.4.8) using (4.3.35). For the 4D calculi we obtain the following four sets of equations

$$\begin{aligned}
&(\mathcal{C}_{st,j}A_{ij} - \mathcal{C}_{jk,i}(A_{js}A_{kt} + B_{js}C_{kt}))\theta_s \wedge \theta_t = \\
&\qquad A_{ij}(\theta_1 \wedge \theta_j + \theta_j \wedge \theta_1) + B_{ij}\theta_j \wedge \theta_3 + C_{ij}\theta_2 \wedge \theta_j, \\
&(\mathcal{C}_{st,j}C_{ij} - \mathcal{C}_{jk,i}(C_{js}A_{kt} + D_{js}C_{kt}))\theta_s \wedge \theta_t = \\
&\qquad C_{ij}(\theta_4 \wedge \theta_j + \theta_j \wedge \theta_1) + D_{ij}\theta_j \wedge \theta_3 + A_{ij}\theta_3 \wedge \theta_j, \\
&(\mathcal{C}_{st,j}B_{ij} - \mathcal{C}_{jk,i}(A_{js}B_{kt} + B_{js}D_{kt}))\theta_s \wedge \theta_t = \\
&\qquad B_{ij}(\theta_1 \wedge \theta_j + \theta_j \wedge \theta_4) + A_{ij}\theta_j \wedge \theta_2 + D_{ij}\theta_2 \wedge \theta_j, \\
&(\mathcal{C}_{st,j}D_{ij} - \mathcal{C}_{jk,i}(C_{js}B_{kt} + D_{js}D_{kt}))\theta_s \wedge \theta_t = \\
&\qquad D_{ij}(\theta_4 \wedge \theta_j + \theta_j \wedge \theta_4) + C_{ij}\theta_j \wedge \theta_2 + B_{ij}\theta_3 \wedge \theta_j,
\end{aligned} \tag{4.4.14}$$

where repeated indices are summed from 1 to 4. In the 3D case the commutation relations are given by

$$\begin{aligned}
&(\mathcal{C}_{st,j}A_{ij} - \mathcal{C}_{jk,i}(A_{js}A_{kt} + B_{js}C_{kt}))\theta_s \wedge \theta_t = \\
&\qquad A_{ij}(\theta_1 \wedge \theta_j + \theta_j \wedge \theta_1) + B_{ij}\theta_j \wedge \theta_3 + C_{ij}\theta_2 \wedge \theta_j, \\
&(\mathcal{C}_{st,j}C_{ij} - \mathcal{C}_{jk,i}(C_{js}A_{kt} + D_{js}C_{kt}))\theta_s \wedge \theta_t = \\
&\qquad C_{ij}(\theta_j \wedge \theta_1 - \theta_1 \wedge \theta_j - 2h\theta_3 \wedge \theta_j) + D_{ij}\theta_j \wedge \theta_3 + A_{ij}\theta_3 \wedge \theta_j, \\
&(\mathcal{C}_{st,j}B_{ij} - \mathcal{C}_{jk,i}(A_{js}B_{kt} + B_{js}D_{kt}))\theta_s \wedge \theta_t = \\
&\qquad B_{ij}(\theta_1 \wedge \theta_j - \theta_j \wedge \theta_1 - 2h\theta_j \wedge \theta_3) + A_{ij}\theta_j \wedge \theta_2 + D_{ij}\theta_2 \wedge \theta_j, \qquad (4.4.15) \\
&(\mathcal{C}_{st,j}D_{ij} - \mathcal{C}_{jk,i}(C_{js}B_{kt} + D_{js}D_{kt})\theta_s \wedge \theta_t = \\
&\qquad -D_{ij}(\theta_1 \wedge \theta_j + \theta_j \wedge \theta_1 + 2h\theta_3 \wedge \theta_j + 2h\theta_j \wedge \theta_3) + C_{ij}\theta_j \wedge \theta_2 + B_{ij}\theta_3 \wedge \theta_j,
\end{aligned}$$

where now repeated indices are summed from 1 to 3.

Recalling that $\mathrm{Tr}_h \Theta$ was the single biinvariant form in the 4D calculi, and also the defining characteristic of Woronowicz's bimodule map Λ, (4.3.30), we obtain a general relation which must be consistent with commutation relations between 1-forms in the 4D calculi:

$$\begin{aligned}
0 = \mathrm{Tr}_h \Theta \wedge \mathrm{Tr}_h \Theta = \theta_1 \wedge \theta_1 &+ 4h^2\theta_3 \wedge \theta_3 + \theta_4 \wedge \theta_4 \\
&+ 2h(\theta_1 \wedge \theta_3 + \theta_3 \wedge \theta_1) + \theta_1 \wedge \theta_4 + \theta_4 \wedge \theta_1 \\
&+ 2h(\theta_3 \wedge \theta_4 + \theta_4 \wedge \theta_3).
\end{aligned} \tag{4.4.16}$$

Further, in any calculi where this bi-invariant form implements the differential in the sense of Remark (4.3.18), so that on arbitrary 1-forms ω we have

$$\mathrm{d}\omega = \frac{1}{\kappa}[Tr_h\Theta, \omega]_+, \tag{4.4.17}$$

where κ is some constant, we will have further relations

$$[Tr_h\Theta, \theta_i] = \kappa \sum_{j,k=1\ldots4} \mathcal{C}_{jk,i}\theta_j \wedge \theta_k. \tag{4.4.18}$$

Again these must be consistent with relations coming from (4.4.14).

The commutator expression for the quantum Lie bracket, (4.3.47), requires that we know the explicit form of the matrix Λ whose components are given in (4.3.31) as $\Lambda_{ij,st} = f_{it}(v_{sk})$. But we know the algebraic elements of the matrix $||v_{ij}||$ and therefore their expressions in terms of the $ABCD$ representation. This is all that is required.

Finally, the structure constants, $C_{ij,k}$, for the quantum Lie bracket, (4.3.45), now follow immediately from (4.3.48) as we know the $\mathcal{C}_{ij,k}$s and $\Lambda_{ij,st}$s.

4.5. 4-dimensional bicovariant calculi on $GL_{h,g}(2)$

We summarise the result of applying the procedure of the previous section to $GL_{h,g}(2)$ in the following theorems.

THEOREM 4.5.1. *There are three 1-parameter families of 4-dimensional first order bicovariant differential calculi on $GL_{h,g}(2)$ whose bimodules of forms are generated as left $GL_{h,g}(2)$-modules by the differentials of the quantum group generators. We will denote the three families by $\Gamma_1^{4\mathrm{D}}$, $\Gamma_2^{4\mathrm{D}}$ and $\Gamma_3^{4\mathrm{D}}$. They are completely characterised by their respective ABCD matrices.*

$\Gamma_1^{4\mathrm{D}}$: *The ABCD matrices are given by*

$$A = \begin{pmatrix} \frac{3z+2}{2} & 0 & \frac{-(3h+g)z-2h}{2} & \frac{-z}{2} \\ h(z+1) & z+1 & -h^2(z+1) & -h(z+1) \\ 0 & 0 & z+1 & 0 \\ \frac{z}{2} & 0 & \frac{(h-g)z+2h}{2} & \frac{z+2}{2} \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & z & h(h+g)(z+1) & 0 \\ 0 & (h+g)(z+1) & -hg(h+g)(z+1) & 0 \\ 0 & 0 & -(h+g)(z+1) & 0 \\ 0 & z & g(h+g)(z+1) & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} \frac{z+2}{2} & 0 & \frac{-(h-g)z+2g}{2} & \frac{z}{2} \\ -g(z+1) & z+1 & -g^2(z+1) & g(z+1) \\ 0 & 0 & z+1 & 0 \\ \frac{-z}{2} & 0 & \frac{-(h+3g)z-2g}{2} & \frac{3z+2}{2} \end{pmatrix}. \tag{4.5.1}$$

Here we must have $z \neq -1$ to ensure the invertibility of $\mathbb{D}$. With $g = h$, the quantum determinant is central in the differential calculus for parameter values $z = 0$ and $z = -2$. The differential of the quantum determinant is

$$\mathrm{d}\mathcal{D} = \frac{z+2}{2}\mathcal{D}\,\mathrm{Tr}_h\,\Theta, \tag{4.5.2}$$

and for $z \neq 0$ the calculi are inner

$$\mathrm{d}a = \frac{1}{2z}[\mathrm{Tr}_h\,\Theta, a], \qquad \mathrm{d}b = \frac{1}{2z}[\mathrm{Tr}_h\,\Theta, b],$$
$$\mathrm{d}c = \frac{1}{2z}[\mathrm{Tr}_h\,\Theta, c], \qquad \mathrm{d}d = \frac{1}{2z}[\mathrm{Tr}_h\,\Theta, d]. \tag{4.5.3}$$

$\Gamma_2^{4\mathrm{D}}$: *The $ABCD$ matrices are given by*

$$A = \begin{pmatrix} \frac{z+2}{2} & 0 & \frac{(h+g)z-2h}{2} & \frac{z}{2} \\ h(z+1) & 1 & h((h+g)z-h) & h(z-1) \\ 0 & 0 & 1 & 0 \\ \frac{-z}{2} & 0 & \frac{-(h+g)z+2h}{2} & \frac{-z+2}{2} \end{pmatrix},$$
$$B = \begin{pmatrix} -hz & 0 & h(h+g)(1-z) & -hz \\ hgz & (h+g) & hg(h+g)(z-1) & ghz \\ z & 0 & (h+g)(z-1) & z \\ -gz & 0 & g(h+g)(1-z) & -gz \end{pmatrix},$$
$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ z & 0 & (h+g)z & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$D = \begin{pmatrix} \frac{-z+2}{2} & 0 & \frac{-(h+g)z+2g}{2} & \frac{-z}{2} \\ g(z-1) & 1 & g((h+g)z-g) & g(z+1) \\ 0 & 0 & 1 & 0 \\ \frac{z}{2} & 0 & \frac{(h+g)z-2g}{2} & \frac{z+2}{2} \end{pmatrix}. \tag{4.5.4}$$

There is no restriction on the value of z in this case. With $g = h$, the quantum determinant is central in the differential calculi for all values of z. The differential of the quantum determinant is

$$\mathrm{d}\mathcal{D} = \frac{2-3z}{2}\mathcal{D}\,\mathrm{Tr}_h\,\Theta, \tag{4.5.5}$$

but the calculi are not inner

$$\begin{aligned} &[\mathrm{Tr}_h\,\Theta, a] = 0, && [\mathrm{Tr}_h\,\Theta, b] = 0, \\ &[\mathrm{Tr}_h\,\Theta, c] = 0, && [\mathrm{Tr}_h\,\Theta, d] = 0. \end{aligned} \tag{4.5.6}$$

$\Gamma_3^{4\mathrm{D}}$: *The $ABCD$ matrices are given by,*

$$A = \begin{pmatrix} \frac{z+2}{2} & 0 & \frac{(h+g)z-2h}{2} & \frac{z}{2} \\ h & 1 & -h^2 & -h \\ 0 & 0 & 1 & 0 \\ \frac{z}{2} & 0 & \frac{(h+g)z+2h}{2} & \frac{z+2}{2} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & h(h+g) & 0 \\ 0 & (h+g) & -hg(h+g) & 0 \\ 0 & 0 & -(h+g) & 0 \\ 0 & 0 & g(h+g) & 0 \end{pmatrix},$$
$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} \frac{z+2}{2} & 0 & \frac{(h+g)z+2g}{2} & \frac{z}{2} \\ -g & 1 & -g^2 & g \\ 0 & 0 & 1 & 0 \\ \frac{z}{2} & 0 & \frac{(h+g)z-2g}{2} & \frac{z+2}{2} \end{pmatrix}. \tag{4.5.7}$$

Like $\Gamma_1^{4\mathrm{D}}$ we must have $z \neq -1$ to ensure the invertibility of $\mathbb{D}$. With $g = h$, the quantum determinant is central in the differential calculus for parameter values $z = 0$ and $z = -2$. The differential of the quantum determinant is

$$\mathrm{d}\mathcal{D} = \frac{z+2}{2}\mathcal{D}\,\mathrm{Tr}_h\,\Theta, \tag{4.5.8}$$

and again the calculi are not inner

$$[\mathrm{Tr}_h \Theta, a] = za\,\mathrm{Tr}_h \Theta, \qquad [\mathrm{Tr}_h \Theta, b] = zb\,\mathrm{Tr}_h \Theta,$$
$$[\mathrm{Tr}_h \Theta, c] = zc\,\mathrm{Tr}_h \Theta, \qquad [\mathrm{Tr}_h \Theta, d] = zd\,\mathrm{Tr}_h \Theta. \tag{4.5.9}$$

For $z = 0$, $\Gamma_1^{4\mathrm{D}} = \Gamma_2^{4\mathrm{D}} = \Gamma_3^{4\mathrm{D}}$, whilst for all other parameter values the calculi are distinct.

PROOF. The matrices were obtained by systematically extracting the implications of the Constraints 1–3. This laborious task is made possible by using the computer algebra package REDUCE [**48**]. REDUCE was also used to check the other results. The results regarding the centrality of the quantum determinant were obtained by investigating under what conditions $\mathbb{D} = I$, since this is precisely the requirement imposed by the relations (4.4.8). To obtain the expressions for the differential of the quantum determinant in the three calculi, we differentiate the expression $\mathcal{D} = ad - bc + hac$, replace on the right hand side differentials by θ_is through (4.4.2), use the commutations relations between the θ_i and the algebra generators provided by the $ABCD$ matrices to commute the left-invariant forms θ_i to the right and finally straighten the algebra coefficients of the θ_i to obtain the quoted results. □

REMARK 4.5.2. Müller-Hoissen [**77**] obtained a corresponding result for first order differential calculi on the standard 2-parameter quantum group, $GL_{q,p}(2)$. In that case, there is just a *single* 1-parameter family of calculi, and *every* member of this family is inner.

Armed with the commutation relations between left-invariant forms and generators provided by the classification of $ABCD$ matrices, we may now obtain commutation relations between the differentials of the generators and the generators. For example, in $\Gamma_1^{4\mathrm{D}}$, starting with $\mathrm{d}aa$, we replace $\mathrm{d}a$ by its expression in terms of left-invariant forms, commute these to the right and then replace the left-invariant forms by their expressions in terms of differentials to obtain

$$\begin{aligned}
\mathrm{d}aa =&(a + hc)\mathrm{d}a + \frac{z}{4}\{9a + (12h - 7g)c + \\
&\mathcal{D}^{-1}\{bac - 3a^2d + 2(h - g)\{4bc^2 - 6adc - (h - 2g)dc^2 + 3h(h - 2g)c^3\}\}\}\mathrm{d}a + \\
&\frac{z}{2}\mathcal{D}^{-1}\{a^2c + (h - g)\{2ac^2 + (h - 2g)c^3\}\}\mathrm{d}b + \\
&(ghc - ha)\mathrm{d}c + \frac{z}{4}\{(7g - 6h)a - ((h - 7g)(4h - 3g) + gh)c + \\
&\mathcal{D}^{-1}\{2ba^2 - 6ha^3 + 3(2h - 3g)a^2d - (2h - 3g)bac + \\
&2(h - g)\{3(h - 6g)adc - 2(h - 6g)bc^2 - 3g(h - 2g)dc^2 + 9gh(h - 2g)c^3\}\}\}\mathrm{d}c \\
&\frac{-z}{2}\mathcal{D}^{-1}\{a^3 + (2h - 3g)a^2c + (h - g)\{(h - 6g)ac^2 - 3g(h - 2g)c^3\}\}\mathrm{d}d.
\end{aligned} \tag{4.5.10}$$

Most of the other commutation relations are much more complicated so we have chosen not to reproduce them here. A feature of this commutation relation, which is shared with the other differential-generator commutation relations for $\Gamma_1^{4\mathrm{D}}$, $\Gamma_2^{4\mathrm{D}}$ and $\Gamma_3^{4\mathrm{D}}$, is that it is not quadratic, but that when $z = 0$, in which case $\Gamma_1^{4\mathrm{D}} = \Gamma_2^{4\mathrm{D}} = \Gamma_3^{4\mathrm{D}}$, it simplifies drastically and does indeed become quadratic. Moreover, when $z = 0$ these quadratic differential-generator commutation relations may be written in R-matrix form as[2]

$$\hat{R}^{-1}\mathrm{d}T_1T_2 = T_1\mathrm{d}T_2\hat{R}. \tag{4.5.11}$$

This R-matrix expression first appeared in the works of Schirrmacher [**86**] and Sudbery [**97**] which were developments on the works of Manin [**73, 74**] and Maltsiniotis [**72**]. These authors treated the co-plane, in our case $\mathbb{A}_{-1}^{2|0}$, as the algebra of differentials $\mathrm{d}x_i$ of the 'coordinates' x_i whose algebra is that of the plane, $\mathbb{A}_1^{2|0}$. They then sought differential calculi expressed in terms of generators T_{ij} and their differentials $\mathrm{d}T_{ij}$ such that the plane and co-plane are invariant under the transformations $x_i \mapsto \sum T_{ij}x_j$ and $\mathrm{d}x_i \mapsto \sum T_{ij}\mathrm{d}x_j + \sum \mathrm{d}T_{ij}x_j$. In the context of Jordanian quantum groups, in [**58**], the author postulated (4.5.11) as the relation defining commutation relations between differentials and generators for $SL_h(2)$.

REMARK 4.5.3. In [**80**], where the differential calculus on the standard quantum group, $GL_{q,p}(2)$, is considered, the authors also observe that the differential-generator commutation relations are not quadratic for general calculi in the 1-parameter family, but that once their free parameter is fixed to zero, quadratic relations are obtained. Moreover in this case they also recover commutation relations with an R-matrix expression.

REMARK 4.5.4. From (4.5.11) it is a simple matter to demonstrate that for the $z = 0$ calculus on $GL_{h,g}(2)$, the commutation relations between the left-invariant forms and the quantum group generators may be written as

$$\Theta_1T_2 = T_2R_{21}\Theta_1R_{12}, \tag{4.5.12}$$

where

$$\Theta = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix}. \tag{4.5.13}$$

Incidentally, this is precisely the relation we obtain if we attempt a naive application of Jurco's construction with, in the notation of [**57**], $\Gamma = \Gamma_1^c \otimes_A \Gamma_1$.

Turning now to the higher order calculi, we have the following result describing commutation relations between the left-invariant forms:

THEOREM 4.5.5. *The commutation relations between the left-invariant forms in the external bicovariant graded algebras $\Omega_1^{4\mathrm{D}}$, $\Omega_2^{4\mathrm{D}}$ and $\Omega_3^{4\mathrm{D}}$ built respectively upon the three families of first order calculi, $\Gamma_1^{4\mathrm{D}}$, $\Gamma_2^{4\mathrm{D}}$ and $\Gamma_3^{4\mathrm{D}}$ using Woronowicz's theory as described in Section 4.3 are as follows*

[2]Of course, $\hat{R}^{-1} = \hat{R}$, but we write it this way to make comparison with the general results of other authors explicit.

Ω_1^{4D}:

$$\begin{aligned}
\theta_3 \wedge \theta_3 &= 0\\
\theta_3 \wedge \theta_4 &= -\frac{2z+1}{z+1}\theta_4 \wedge \theta_3 + \frac{z}{z+1}\theta_1 \wedge \theta_3,\\
\theta_3 \wedge \theta_1 &= -\frac{1}{z+1}\theta_1 \wedge \theta_3 - \frac{z}{z+1}\theta_4 \wedge \theta_3,\\
\theta_3 \wedge \theta_2 &= -\theta_2 \wedge \theta_3 + (h+g)\theta_1 \wedge \theta_3 - (h+g)\theta_4 \wedge \theta_3,\\
\theta_4 \wedge \theta_4 &= \frac{z}{z+1}\theta_2 \wedge \theta_3 - \frac{z(h+g)}{z+1}\theta_1 \wedge \theta_3 + \frac{z(h+g)}{z+1}\theta_4 \wedge \theta_3,\\
\theta_4 \wedge \theta_1 &= -\theta_1 \wedge \theta_4 - \frac{z(h+g)}{z+1}\theta_1 \wedge \theta_3 + \frac{z(h+g)}{z+1}\theta_4 \wedge \theta_3,\\
\theta_4 \wedge \theta_2 &= -\frac{2z+1}{z+1}\theta_2 \wedge \theta_4 + \frac{z}{z+1}\theta_2 \wedge \theta_1 + \frac{(2z+1)(h+g)}{z+1}\theta_2 \wedge \theta_3\\
&\quad - (h+g)^2\theta_1 \wedge \theta_3 + (h+g)^2\theta_4 \wedge \theta_3,\\
\theta_1 \wedge \theta_1 &= -\frac{z}{z+1}\theta_2 \wedge \theta_3,\\
\theta_1 \wedge \theta_2 &= -\frac{1}{z+1}\theta_2 \wedge \theta_1 - \frac{z}{z+1}\theta_2 \wedge \theta_4 - \frac{(h+g)}{z+1}\theta_2 \wedge \theta_3,\\
\theta_2 \wedge \theta_2 &= (h+g)\theta_1 \wedge \theta_2 - (h+g)\theta_2 \wedge \theta_4 + (h+g)^2\theta_2 \wedge \theta_3,
\end{aligned} \tag{4.5.14}$$

Ω_2^{4D}:

$$\begin{aligned}
\theta_3 \wedge \theta_3 &= 0\\
\theta_3 \wedge \theta_4 &= -\theta_4 \wedge \theta_3,\\
\theta_3 \wedge \theta_1 &= -\theta_1 \wedge \theta_3,\\
\theta_3 \wedge \theta_2 &= -\theta_2 \wedge \theta_3 + (h+g)\theta_1 \wedge \theta_3 - (h+g)\theta_4 \wedge \theta_3,\\
\theta_4 \wedge \theta_4 &= 0,\\
\theta_4 \wedge \theta_1 &= -\theta_1 \wedge \theta_4,\\
\theta_4 \wedge \theta_2 &= -\theta_2 \wedge \theta_4 + (h+g)\theta_2 \wedge \theta_3 - (h+g)^2\theta_1 \wedge \theta_3 + (h+g)^2\theta_4 \wedge \theta_3,\\
\theta_1 \wedge \theta_1 &= 0,\\
\theta_1 \wedge \theta_2 &= -\theta_2 \wedge \theta_1 - (h+g)\theta_2 \wedge \theta_3,\\
\theta_2 \wedge \theta_2 &= (h+g)\theta_1 \wedge \theta_2 - (h+g)\theta_2 \wedge \theta_4 + (h+g)^2\theta_2 \wedge \theta_3,
\end{aligned} \tag{4.5.15}$$

$\Omega_3^{4\mathrm{D}}$:

$$
\begin{aligned}
&\theta_3 \wedge \theta_3 = 0\\
&\theta_3 \wedge \theta_4 = -\theta_4 \wedge \theta_3,\\
&\theta_3 \wedge \theta_1 = -\theta_1 \wedge \theta_3,\\
&\theta_3 \wedge \theta_2 = -\theta_2 \wedge \theta_3 + (h+g)\theta_1 \wedge \theta_3 - (h+g)\theta_4 \wedge \theta_3,\\
&\theta_4 \wedge \theta_4 = 0,\\
&\theta_4 \wedge \theta_1 = -\theta_1 \wedge \theta_4,\\
&\theta_4 \wedge \theta_2 = -\theta_2 \wedge \theta_4 + (h+g)\theta_2 \wedge \theta_3 - (h+g)^2\theta_1 \wedge \theta_3 + (h+g)^2\theta_4 \wedge \theta_3,\\
&\theta_1 \wedge \theta_1 = 0,\\
&\theta_1 \wedge \theta_2 = -\theta_2 \wedge \theta_1 - (h+g)\theta_2 \wedge \theta_3,\\
&\theta_2 \wedge \theta_2 = (h+g)\theta_1 \wedge \theta_2 - (h+g)\theta_2 \wedge \theta_4 + (h+g)^2\theta_2 \wedge \theta_3. \qquad (4.5.16)
\end{aligned}
$$

The relations in each case are consistent with the respective relations (4.4.8). They are also consistent with the relation (4.4.16) and in the case of $\Omega_1^{4\mathrm{D}}$, the relations (4.4.18). Further, they are such that $\{\theta_2^\alpha\theta_1^\beta\theta_4^\gamma\theta_3^\delta : \alpha,\beta,\gamma,\delta \in \{0,1\}\}$ is a basis for the exterior algebra of forms in each case.

PROOF. Once again we used REDUCE to check these results based on the discussion of Section 4.4. The 16 equations (4.4.14) were treated, in each of the cases, $\Gamma_1^{4\mathrm{D}}$, $\Gamma_2^{4\mathrm{D}}$ and $\Gamma_3^{4\mathrm{D}}$ linear relations between the 10 'mis-ordered' 2-forms, $\theta_3 \wedge \theta_3$, $\theta_3 \wedge \theta_4$, $\theta_3 \wedge \theta_1$, $\theta_3 \wedge \theta_2$, $\theta_4 \wedge \theta_4$, $\theta_4 \wedge \theta_1$, $\theta_4 \wedge \theta_2$, $\theta_1 \wedge \theta_1$, $\theta_1 \wedge \theta_2$ and $\theta_2 \wedge \theta_2$, and the 6 'ordered' 2-forms, $\theta_2 \wedge \theta_1$, $\theta_2 \wedge \theta_4$, $\theta_2 \wedge \theta_3$, $\theta_1 \wedge \theta_4$, $\theta_1 \wedge \theta_3$ and $\theta_4 \wedge \theta_3$. These linear relations were then solved for the 10 mis-ordered 2-forms yielding in each case the single solution presented. Consistency with the relations (4.4.8) in each of the three cases was checked by commuting the generators through the relations and observing that no further conditions were incurred. The other consistency conditions were again checked by direct computation. Finally, observing that the relations are compatible with the ordering $\theta_2 \prec \theta_1 \prec \theta_4 \prec \theta_3$, we may use the Diamond Lemma to prove the statement about the bases. □

REMARK 4.5.6. It is interesting to note here that the relations in $\Gamma_2^{4\mathrm{D}}$ and $\Gamma_3^{4\mathrm{D}}$ are the same and indeed could be obtained from those in $\Gamma_1^{4\mathrm{D}}$ by setting $z = 0$.

REMARK 4.5.7. In contrast with the results in Theorem 4.5.5, in the work of Müller-Hoissen and Reuten on $GL_{q,p}(2)$ [**80**] the corresponding commutation relations between left-invariant forms exhibited ordering circles which introduced further constraints on the free parameter of their family of calculi.

Let us now describe the quantum Lie brackets in the quantum Lie algebras $\mathcal{L}_1^{4\mathrm{D}}$, $\mathcal{L}_2^{4\mathrm{D}}$ and $\mathcal{L}_3^{4\mathrm{D}}$ dual to the bicovariant bimodules $\Gamma_1^{4\mathrm{D}}$, $\Gamma_2^{4\mathrm{D}}$ and $\Gamma_3^{4\mathrm{D}}$ respectively.

THEOREM 4.5.8. *The quantum Lie brackets and quantum commutators for the quantum Lie algebras $\mathcal{L}_1^{4\mathrm{D}}$, $\mathcal{L}_2^{4\mathrm{D}}$ and $\mathcal{L}_3^{4\mathrm{D}}$ as described in Section 4.3 are as follows:*

$\mathcal{L}_1^{4\mathrm{D}}$: *The bracket relations are*

$$
\begin{aligned}
[\chi_1,\chi_1] &= 0,\\
[\chi_1,\chi_2] &= \frac{1}{z+1}\chi_2,\\
[\chi_1,\chi_3] &= -\frac{1}{z+1}\chi_3+\frac{h+g}{z+1}\chi_4,\\
[\chi_1,\chi_4] &= 0,\\
[\chi_2,\chi_1] &= -\frac{1}{z+1}\chi_2,\\
[\chi_2,\chi_2] &= 0,\\
[\chi_2,\chi_3] &= \frac{1}{z+1}\chi_1-\frac{h+g}{z+1}\chi_2-\frac{1}{z+1}\chi_4,\\
[\chi_2,\chi_4] &= \frac{1}{z+1}\chi_2,\\
[\chi_3,\chi_1] &= -\frac{h+g}{z+1}\chi_1+\frac{1}{z+1}\chi_3,\\
[\chi_3,\chi_2] &= -\frac{1}{z+1}\chi_1-\frac{h+g}{z+1}\chi_2+\frac{1}{z+1}\chi_4,\\
[\chi_3,\chi_3] &= \frac{(h+g)^2}{z+1}\chi_1-2\frac{h+g}{z+1}\chi_3+\frac{(h+g)^2}{z+1}\chi_4,\\
[\chi_3,\chi_4] &= \frac{h+g}{z+1}\chi_1-\frac{1}{z+1}\chi_3,\\
[\chi_4,\chi_1] &= 0,\\
[\chi_4,\chi_2] &= -\frac{1}{z+1}\chi_2,\\
[\chi_4,\chi_3] &= \frac{1}{z+1}\chi_3-\frac{h+g}{z+1}\chi_4,\\
[\chi_4,\chi_4] &= 0,
\end{aligned}
\tag{4.5.17}
$$

and the commutators are

$$
\begin{aligned}
[\chi_1,\chi_1] &= 0,\\
[\chi_1,\chi_2] &= \chi_1\chi_2-\Big(\frac{z}{z+1}\chi_1\chi_2+\chi_2\chi_1+(h+g)\chi_2\chi_2+\frac{z}{z+1}\chi_4\chi_2\Big),\\
[\chi_1,\chi_3] &= \chi_1\chi_3-\Big(\frac{z}{z+1}\chi_1\chi_3+\frac{z(h+g)}{z+1}\chi_1\chi_4+\chi_3\chi_1-(h+g)\chi_3\chi_2+(h+g)^2\chi_4\chi_2\\
&\qquad -\frac{z}{z+1}\chi_4\chi_3+\frac{z(h+g)}{z+1}\chi_4\chi_4\Big),\\
[\chi_1,\chi_4] &= \chi_1\chi_4-\chi_4\chi_1,\\
[\chi_2,\chi_1] &= \chi_2\chi_1-\Big(\frac{1}{z+1}\chi_1\chi_2-(h+g)\chi_2\chi_2-\frac{z}{z+1}\chi_4\chi_2\Big),\\
[\chi_2,\chi_2] &= 0,
\end{aligned}
$$

$$
\begin{aligned}
[\chi_2, \chi_3] &= \chi_2\chi_3 - \Big(\frac{z}{z+1}\chi_1\chi_1 + \frac{h+g}{z+1}\chi_1\chi_2 - \frac{z}{z+1}\chi_1\chi_4 - (h+g)^2\chi_2\chi_2 + \chi_3\chi_2 \\
&\quad + \frac{z}{z+1}\chi_4\chi_1 - \frac{(h+g)(2z+1)}{z+1}\chi_4\chi_2 - \frac{z}{z+1}\chi_4\chi_4\Big), \\
[\chi_2, \chi_4] &= \chi_2\chi_4 - \Big(\frac{z}{z+1}\chi_1\chi_2 + (h+g)\chi_2\chi_2 + \frac{2z+1}{z+1}\chi_4\chi_2\Big), \\
[\chi_3, \chi_1] &= \chi_3\chi_1 - \Big(-\frac{z(h+g)}{z+1}\chi_1\chi_1 + \frac{2z+1}{z+1}\chi_1\chi_3 - (h+g)^2\chi_2\chi_1 + (h+g)\chi_2\chi_3 \\
&\quad - \frac{z(h+g)}{z+1}\chi_4\chi_1 + \frac{z}{z+1}\chi_4\chi_3\Big), \\
[\chi_3, \chi_2] &= \chi_3\chi_2 - \Big(-\frac{z}{z+1}\chi_1\chi_1 - \frac{z(h+g)}{z+1}\chi_1\chi_2 + \frac{z}{z+1}\chi_1\chi_4 - (h+g)\chi_2\chi_1 \\
&\quad - (h+g)^2\chi_2\chi_2 + \chi_2\chi_3 + (h+g)\chi_2\chi_4 - \frac{z}{z+1}\chi_4\chi_1 - \frac{z(h+g)}{z+1}\chi_4\chi_2 \\
&\quad + \frac{z}{z+1}\chi_4\chi_4\Big), \\
[\chi_3, \chi_3] &= \chi_3\chi_3 - \Big(\frac{z(h+g)^2}{z+1}\chi_1\chi_1 - \frac{(h+g)(3z+1)}{z+1}\chi_1\chi_3 + \frac{(h+g)^2(2z+1)}{z+1}\chi_1\chi_4 \\
&\quad + (h+g)^3\chi_2\chi_1 - (h+g)^2\chi_2\chi_3 + (h+g)\chi_3\chi_1 - (h+g)^2\chi_3\chi_2 + \chi_3\chi_3 \\
&\quad - (h+g)\chi_3\chi_4 - \frac{(h+g)^2}{z+1}\chi_4\chi_1 + (h+g)^3\chi_4\chi_2 - \frac{(h+g)(z-1)}{z+1}\chi_4\chi_3 \\
&\quad + \frac{z(h+g)^2}{z+1}\chi_4\chi_4\Big), \\
[\chi_3, \chi_4] &= \chi_3\chi_4 - \Big(\frac{z(h+g)}{z+1}\chi_1\chi_1 - \frac{z}{z+1}\chi_1\chi_3 + (h+g)^2\chi_2\chi_1 - (h+g)\chi_2\chi_3 \\
&\quad + \frac{z(h+g)}{z+1}\chi_4\chi_1 + \frac{1}{z+1}\chi_4\chi_3\Big), \\
[\chi_4, \chi_1] &= \chi_4\chi_1 - \chi_1\chi_4, \\
[\chi_4, \chi_2] &= \chi_4\chi_2 - \Big(-\frac{z}{z+1}\chi_1\chi_2 - (h+g)\chi_2\chi_2 + \chi_2\chi_4 - \frac{z}{z+1}\chi_4\chi_2\Big), \\
[\chi_4, \chi_3] &= \chi_4\chi_3 - \Big(\frac{z}{z+1}\chi_1\chi_3 - \frac{z(h+g)}{z+1}\chi_1\chi_4 + (h+g)\chi_3\chi_2 + \chi_3\chi_4 - (h+g)^2\chi_4\chi_2 \\
&\quad + \frac{z}{z+1}\chi_4\chi_3 - \frac{z(h+g)}{z+1}\chi_4\chi_4\Big), \\
[\chi_4, \chi_4] &= 0.
\end{aligned}
\tag{4.5.18}
$$

$\mathcal{L}_2^{4\mathrm{D}}$: *The bracket relations are*

$$
\begin{aligned}
[\chi_1, \chi_1] &= z\chi_1 - z\chi_4, \\
[\chi_1, \chi_2] &= \chi_2, \\
[\chi_1, \chi_3] &= z(h+g)\chi_1 - \chi_3 - (z-1)(h+g)\chi_4, \\
[\chi_1, \chi_4] &= z\chi_1 - z\chi_4, \\
[\chi_2, \chi_1] &= (2z-1)\chi_2,
\end{aligned}
$$

$$\begin{aligned}
&[\chi_2,\chi_2]=0,\\
&[\chi_2,\chi_3]=\chi_1+(2z-1)(h+g)\chi_2-\chi_4,\\
&[\chi_2,\chi_4]=(2z+1)\chi_2,\\
&[\chi_3,\chi_1]=-(z+1)(h+g)\chi_1+(2z+1)\chi_3-z(h+g)\chi_4,\\
&[\chi_3,\chi_2]=-\chi_1-(h+g)\chi_2+\chi_4,\\
&[\chi_3,\chi_3]=-(z-1)(h+g)^2\chi_1+2(z-1)(h+g)\chi_3-(z-1)(h+g)^2\chi_4,\\
&[\chi_3,\chi_4]=-(z-1)(h+g)\chi_1+(2z-1)\chi_3-z(h+g)\chi_4,\\
&[\chi_4,\chi_1]=-z\chi_1+z\chi_4,\\
&[\chi_4,\chi_2]=-\chi_2,\\
&[\chi_4,\chi_3]=-z(h+g)\chi_1+\chi_3+(z-1)(h+g)\chi_4,\\
&[\chi_4,\chi_4]=-z\chi_1+z\chi_4,
\end{aligned}\tag{4.5.19}$$

and the commutators are

$$\begin{aligned}
[\chi_1,\chi_1]&=\chi_1\chi_1-\Big(\chi_1\chi_1+z(h+g)\chi_2\chi_1-z\chi_2\chi_3+z\chi_3\chi_2-z(h+g)\chi_4\chi_2\Big),\\
[\chi_1,\chi_2]&=\chi_1\chi_2-\Big(\chi_2\chi_1+(h+g)\chi_2\chi_2\Big),\\
[\chi_1,\chi_3]&=\chi_1\chi_3-\Big(z(h+g)^2\chi_2\chi_1-z(h+g)\chi_2\chi_3+\chi_3\chi_1+(h+g)(z-1)\chi_3\chi_2\\
&\qquad-(h+g)^2(z-1)\chi_4\chi_2\Big),\\
[\chi_1,\chi_4]&=\chi_1\chi_4-\Big(z(h+g)\chi_2\chi_1-z\chi_2\chi_3+z\chi_3\chi_2+\chi_4\chi_1-z(h+g)\chi_4\chi_2\Big),\\
[\chi_2,\chi_1]&=\chi_2\chi_1-\Big(-(z-1)\chi_1\chi_2+z\chi_2\chi_1+(h+g)(2z-1)\chi_2\chi_2-z\chi_2\chi_4\\
&\qquad+z\chi_4\chi_2\Big),\\
[\chi_2,\chi_2]&=0,\\
[\chi_2,\chi_3]&=\chi_2\chi_3-\Big(-(h+g)(z-1)\chi_1\chi_2+z(h+g)\chi_2\chi_1+(h+g)^2(2z-1)\chi_2\chi_2\\
&\qquad-z(h+g)\chi_2\chi_4+\chi_3\chi_2+(h+g)(z-1)\chi_4\chi_2\Big),\\
[\chi_2,\chi_4]&=\chi_2\chi_4-\Big(-z\chi_1\chi_2+z\chi_2\chi_1+(h+g)(2z+1)\chi_2\chi_2-z\chi_2\chi_4+(z+1)\chi_4\chi_2\Big),\\
[\chi_3,\chi_1]&=\chi_3\chi_1-\Big((z+1)\chi_1\chi_3-z(h+g)\chi_1\chi_4-(h+g)^2(z+1)\chi_2\chi_1+(h+g)(z+1)\chi_2\chi_3\\
&\qquad-z\chi_3\chi_1+z(h+g)\chi_3\chi_2+z\chi_3\chi_4+z(h+g)\chi_4\chi_1-z(h+g)^2\chi_4\chi_2-z\chi_4\chi_3\Big),\\
[\chi_3,\chi_2]&=\chi_3\chi_2-\Big(-(h+g)\chi_2\chi_1-(h+g)^2\chi_2\chi_2+\chi_2\chi_3+(h+g)\chi_2\chi_4\Big),\\
[\chi_3,\chi_3]&=\chi_3\chi_3-\Big((h+g)(z-1)\chi_1\chi_3-(h+g)^2(z-1)\chi_1\chi_4-(h+g)^3(z-1)\chi_2\chi_1
\end{aligned}$$

$$
\begin{aligned}
&+ (h+g)^2(z-1)\chi_2\chi_3 - (h+g)(z-1)\chi_3\chi_1 + (h+g)^2(z-1)\chi_3\chi_2 + \chi_3\chi_3 \\
&+ (h+g)(z-1)\chi_3\chi_4 + (h+g)^2(z-1)\chi_4\chi_1 - (h+g)^3(z-1)\chi_4\chi_2 \\
&- (h+g)(z-1)\chi_4\chi_3\Big), \\
[\chi_3,\chi_4] &= \chi_3\chi_4 - \Big(z\chi_1\chi_3 - z(h+g)\chi_1\chi_4 - (h+g)^2(z-1)\chi_2\chi_1 + (h+g)(z-1)\chi_2\chi_3 \\
&- z\chi_3\chi_1 + z(h+g)\chi_3\chi_2 + z\chi_3\chi_4 + z(h+g)\chi_4\chi_1 - z(h+g)^2\chi_4\chi_2 \\
&- (z-1)\chi_4\chi_3\Big), \\
[\chi_4,\chi_1] &= \chi_4\chi_1 - \Big(\chi_1\chi_4 - z(h+g)\chi_2\chi_1 + z\chi_2\chi_3 - z\chi_3\chi_2 + z(h+g)\chi_4\chi_2\Big), \\
[\chi_4,\chi_2] &= \chi_4\chi_2 - \Big(-(h+g)\chi_2\chi_2 + \chi_2\chi_4\Big), \\
[\chi_4,\chi_3] &= \chi_4\chi_3 - \Big(-z(h+g)^2 + z(h+g)\chi_2\chi_3 - (h+g)(z-1)\chi_3\chi_2 + \chi_3\chi_4 \\
&+ (h+g)^2(z-1)\chi_4\chi_2\Big), \\
[\chi_4,\chi_4] &= \chi_4\chi_4 - \Big(-z(h+g)\chi_2\chi_1 + z\chi_2\chi_3 - z\chi_3\chi_2 + z(h+g)\chi_4\chi_2 + \chi_4\chi_4\Big). \qquad (4.5.20)
\end{aligned}
$$

$\mathcal{L}_3^{4\mathrm{D}}$: *The bracket relations are*

$$
\begin{aligned}
[\chi_1,\chi_1] &= 0, \\
[\chi_1,\chi_2] &= \chi_2, \\
[\chi_1,\chi_3] &= -\chi_3 + (h+g)\chi_4, \\
[\chi_1,\chi_4] &= 0, \\
[\chi_2,\chi_1] &= -\chi_2, \\
[\chi_2,\chi_2] &= 0, \\
[\chi_2,\chi_3] &= \chi_1 - (h+g)\chi_2 - \chi_4, \\
[\chi_2,\chi_4] &= \chi_2, \\
[\chi_3,\chi_1] &= -(h+g)\chi_1 + \chi_3, \\
[\chi_3,\chi_2] &= -\chi_1 - (h+g)\chi_2 + \chi_4, \\
[\chi_3,\chi_3] &= (h+g)^2\chi_1 - 2(h+g)\chi_3 + (h+g)^2\chi_4, \\
[\chi_3,\chi_4] &= (h+g)\chi_1 - \chi_3, \\
[\chi_4,\chi_1] &= 0, \\
[\chi_4,\chi_2] &= -\chi_2, \\
[\chi_4,\chi_3] &= \chi_3 - (h+g)\chi_4, \\
[\chi_4,\chi_4] &= 0, \qquad (4.5.21)
\end{aligned}
$$

and the commutators are

$$[\chi_1,\chi_1] = 0,$$

$$
\begin{aligned}
[\chi_1, \chi_2] &= \chi_1\chi_2 - \Big(\chi_2\chi_1 + (h+g)\chi_2\chi_2\Big),\\
[\chi_1, \chi_3] &= \chi_1\chi_3 - \Big(\chi_3\chi_1 - (h+g)\chi_3\chi_2 + (h+g)^2\chi_4\chi_2\Big),\\
[\chi_1, \chi_4] &= 0,\\
[\chi_2, \chi_1] &= \chi_2\chi_1 - \Big(\chi_1\chi_2 - (h+g)\chi_2\chi_2\Big),\\
[\chi_2, \chi_2] &= 0,\\
[\chi_2, \chi_3] &= \chi_2\chi_3 - \Big((h+g)\chi_1\chi_2 - (h+g)^2\chi_2\chi_2 + \chi_3\chi_2 - (h+g)\chi_4\chi_2\Big),\\
[\chi_2, \chi_4] &= \chi_2\chi_4 - \Big((h+g)\chi_2\chi_2 + \chi_4\chi_2\Big),\\
[\chi_3, \chi_1] &= \chi_3\chi_1 - \Big(\chi_1\chi_3 - (h+g)^2\chi_2\chi_1 + (h+g)\chi_2\chi_3\Big),\\
[\chi_3, \chi_2] &= \chi_3\chi_2 - \Big(-(h+g)\chi_2\chi_1 - (h+g)^2\chi_2\chi_2 + \chi_2\chi_3 + (h+g)\chi_2\chi_4\Big),\\
[\chi_3, \chi_3] &= \chi_3\chi_3 - \Big(-(h+g)\chi_1\chi_3 + (h+g)^2\chi_1\chi_4 + (h+g)^3\chi_2\chi_1 - (h+g)^2\chi_2\chi_3\\
&\qquad + (h+g)\chi_3\chi_1 - (h+g)^2\chi_3\chi_2 + \chi_3\chi_3 - (h+g)\chi_3\chi_4 - (h+g)^2\chi_4\chi_1\\
&\qquad + (h+g)^3\chi_4\chi_2 + (h+g)\chi_4\chi_3\Big),\\
[\chi_3, \chi_4] &= \chi_3\chi_4 - \Big((h+g)^2\chi_2\chi_1 - (h+g)\chi_2\chi_3 + \chi_4\chi_3\Big),\\
[\chi_4, \chi_1] &= 0,\\
[\chi_4, \chi_2] &= \chi_4\chi_2 - \Big(-(h+g)\chi_2\chi_2 + \chi_2\chi_4\Big),\\
[\chi_4, \chi_3] &= \chi_4\chi_3 - \Big((h+g)\chi_3\chi_2 + \chi_3\chi_4 - (h+g)^2\chi_4\chi_2\Big),\\
[\chi_4, \chi_4] &= 0. \qquad (4.5.22)
\end{aligned}
$$

The following result reveals that the relations in the universal enveloping algebras $U(\mathcal{L}_1^{4\mathrm{D}})$, $U(\mathcal{L}_2^{4\mathrm{D}})$ and $U(\mathcal{L}_3^{4\mathrm{D}})$ reflect the structure of the commutation relations of the left-invariant forms presented in Theorem 4.5.5. Indeed the relations in $U(\mathcal{L}_2^{4\mathrm{D}})$ and $U(\mathcal{L}_3^{4\mathrm{D}})$ are identical and can be obtained from those in $U(\mathcal{L}_1^{4\mathrm{D}})$ by setting $z = 0$.

THEOREM 4.5.9. *The relations in the universal enveloping algebras, $U(\mathcal{L}_1^{4\mathrm{D}})$, $U(\mathcal{L}_2^{4\mathrm{D}})$ and $U(\mathcal{L}_3^{4\mathrm{D}})$ are as follows:*

$U(\mathcal{L}_1^{4\mathrm{D}})$:

$$
\begin{aligned}
\chi_3\chi_4 = \frac{z(h+g)}{z+1}\chi_1^2 + (h+g)^2\chi_2\chi_1 - (h+g)\chi_2\chi_3 + \frac{z(h+g)}{z+1}\chi_1\chi_4 - \frac{z}{z+1}\chi_1\chi_3\\
+ \frac{1}{z+1}\chi_4\chi_3 + \frac{h+g}{z+1}\chi_1 - \frac{1}{z+1}\chi_3,
\end{aligned}
$$

$$\begin{aligned}
\chi_3\chi_1 &= -\frac{z(h+g)}{z+1}\chi_1^2 - (h+g)^2\chi_2\chi_1 + (h+g)\chi_2\chi_3 - \frac{z(h+g)}{z+1}\chi_1\chi_4 + \frac{2z+1}{z+1}\chi_1\chi_3 \\
&\quad + \frac{z}{z+1}\chi_4\chi_3 - \frac{h+g}{z+1}\chi_1 + \frac{1}{z+1}\chi_3, \\
\chi_3\chi_2 &= \frac{z}{z+1}\chi_4^2 - \frac{z}{z+1}\chi_1^2 - (h+g)^2\chi_2^2 - \frac{(2z+1)(h+g)}{z+1}\chi_2\chi_1 + \frac{h+g}{z+1}\chi_2\chi_4 + \chi_2\chi_3 \\
&\quad - \frac{1}{z+1}\chi_1 - \frac{h+g}{z+1}\chi_2 + \frac{1}{z+1}\chi_4, \\
\chi_4\chi_1 &= \chi_1\chi_4, \\
\chi_4\chi_2 &= -(h+g)\chi_2^2 - \frac{z}{z+1}\chi_2\chi_1 + \frac{1}{z+1}\chi_2\chi_4 - \frac{1}{z+1}\chi_2, \\
\chi_1\chi_2 &= (h+g)\chi_2^2 + \frac{2z+1}{z+1}\chi_2\chi_1 + \frac{z}{z+1}\chi_2\chi_4 + \frac{1}{z+1}\chi_2.
\end{aligned} \tag{4.5.23}$$

$U(\mathcal{L}_2^{4\mathrm{D}})$:

$$\begin{aligned}
\chi_3\chi_4 &= (h+g)^2\chi_2\chi_1 - (h+g)\chi_2\chi_3 + \chi_4\chi_3 + (h+g)\chi_1 - \chi_3, \\
\chi_3\chi_1 &= -(h+g)^2\chi_2\chi_1 + (h+g)\chi_2\chi_3 + \chi_1\chi_3 - (h+g)\chi_1 + \chi_3, \\
\chi_3\chi_2 &= -(h+g)^2\chi_2^2 - (h+g)\chi_2\chi_1 + (h+g)\chi_2\chi_4 + \chi_2\chi_3 - \chi_1 - (h+g)\chi_2 + \chi_4, \\
\chi_4\chi_1 &= \chi_1\chi_4, \\
\chi_4\chi_2 &= -(h+g)\chi_2^2 + \chi_2\chi_4 - \chi_2, \\
\chi_1\chi_2 &= (h+g)\chi_2^2 + \chi_2\chi_1 + \chi_2.
\end{aligned} \tag{4.5.24}$$

$U(\mathcal{L}_3^{4\mathrm{D}})$:

$$\begin{aligned}
\chi_3\chi_4 &= (h+g)^2\chi_2\chi_1 - (h+g)\chi_2\chi_3 + \chi_4\chi_3 + (h+g)\chi_1 - \chi_3, \\
\chi_3\chi_1 &= -(h+g)^2\chi_2\chi_1 + (h+g)\chi_2\chi_3 + \chi_1\chi_3 - (h+g)\chi_1 + \chi_3, \\
\chi_3\chi_2 &= -(h+g)^2\chi_2^2 - (h+g)\chi_2\chi_1 + (h+g)\chi_2\chi_4 + \chi_2\chi_3 - \chi_1 - (h+g)\chi_2 + \chi_4, \\
\chi_4\chi_1 &= \chi_1\chi_4, \\
\chi_4\chi_2 &= -(h+g)\chi_2^2 + \chi_2\chi_4 - \chi_2, \\
\chi_1\chi_2 &= (h+g)\chi_2^2 + \chi_2\chi_1 + \chi_2.
\end{aligned} \tag{4.5.25}$$

In each case the relations are such that $\{\chi_2^\alpha\chi_1^\beta\chi_4^\gamma\chi_2^\delta : \alpha,\beta,\gamma,\delta \in \mathbb{Z}_{\geq 0}\}$ *is a basis of the enveloping algebra.*

Proof. These relations are obtained by solving the 16 equations

$$\chi_i\chi_k = \sum_{s,t=1\ldots d} \Lambda_{st,ik}\chi_s\chi_t + \sum_{j=1\ldots d} C_{ik,j}\chi_j, \tag{4.5.26}$$

for the 6 quadratic elements $\chi_3\chi_4$, $\chi_3\chi_1$, $\chi_3\chi_2$, $\chi_4\chi_1$, $\chi_4\chi_2$ and $\chi_1\chi_2$. It is then observed that the relations are compatible with the ordering $\chi_2 \prec \chi_1 \prec \chi_4 \prec \chi_3$ so the Diamond Lemma may be applied to obtain the stated basis. $\square$

4.6. 3-dimensional bicovariant differential calculi on $SL_h(2)$

Classically (see for example the discussion in the book by Flanders [**40**]) we obtain the differential calculus on $SL(2)$ from the calculus on $GL(2)$ through the classical relation,

$$\mathrm{d}\mathcal{D} = \mathcal{D}\,\mathrm{Tr}\,\Theta^c, \tag{4.6.1}$$

where Θ^c is the classical matrix of left-invariant 1-forms. $SL(2)$ and its differential calculus is obtained by setting $\mathcal{D} = 1$, so the left hand side becomes zero and we obtain a linear relation between the classical left-invariant 1-forms, namely, $\theta_1 + \theta_4 = 0$. In the standard quantum $GL_q(2)$ case this procedure is *not* possible. The analogue of (4.6.1) in this case is rendered trivial by the condition that $\mathcal{D}$ be central in the first order calculus. More precisely, we have in this case

$$\mathrm{d}\mathcal{D} = \kappa\mathcal{D}\,\mathrm{Tr}_q\,\Theta, \tag{4.6.2}$$

where $\mathrm{Tr}_q\,\Theta$ is now the q-analogue of $\mathrm{Tr}\,\Theta$, $\theta_1 + q^{-1}\theta_4$. But now, imposing the condition $\mathcal{D} = 1$, immediately fixes $\kappa = 0$ so we have no chance of reducing the dimension of the calculus. There are of course 4-dimensional bicovariant calculi on the standard quantum group $SL_q(2)$ but *no* 3-dimensional calculi.

In our case, with the non-standard Jordanian quantum group, the situation is quite different. Studying Theorem 4.5.1 we see that for the calculi $\Gamma_1^{4\mathrm{D}}$ and $\Gamma_3^{4\mathrm{D}}$ we can indeed have the quantum determinant central *and* obtain a dimension reducing relation through the analogue in these cases of (4.6.1)

$$\mathrm{d}\mathcal{D} = \frac{z+2}{2}\mathcal{D}\,\mathrm{Tr}_h\,\Theta. \tag{4.6.3}$$

The quantum determinant is central in $\Gamma_1^{4\mathrm{D}}$ and $\Gamma_3^{4\mathrm{D}}$ if and only if the parameter z takes the value 0 or -2. With $z = -2$ we obtain in each case a 4-dimensional calculus on $SL_h(2)$, while with $z = 0$ the condition $\mathrm{d}\mathcal{D} = 0$ yields the linear relation $\mathrm{Tr}_h\,\Theta = \theta_1 + 2h\theta_3 + \theta_4 = 0$ — precisely the relation (4.4.6) we obtained when we investigated the general implications of choosing a three dimensional basis of Γ_{inv}. Moreover, $z = 0$ is the value of z at which $\Gamma_1^{4\mathrm{D}}$, $\Gamma_2^{4\mathrm{D}}$ and $\Gamma_3^{4\mathrm{D}}$ coincide, so is already covered as a particular case in $\Gamma_2^{4\mathrm{D}}$. For this first order calculus, having set $g = h$, the quantum determinant is central for *all* values of z. In the particular case of $z = \frac{2}{3}$, the differential of the quantum determinant is identically zero so that once again we obtain a 4-dimensional calculus on $SL_h(2)$. However, for all other values of z we recover the condition $\mathrm{Tr}_h\,\Theta = 0$. At first sight then, it may seem that there is a family of 3-dimensional calculi on $SL_h(2)$, but this is not the case.

THEOREM 4.6.1. *There is a unique, 3-dimensional, first order bicovariant differential calculus on the Jordanian quantum group $SL_h(2)$, $\Gamma^{3\mathrm{D}}$. It may be obtained from any one of the three families of first order bicovariant differential calculi on $GL_{h,g}(2)$ by a reduction analogous to the classical situation. It is specified by its $ABCD$ matrices:*

$$A = \begin{pmatrix} 1 & 0 & -h \\ 2h & 1 & h^2 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 2h^2 \\ 0 & 2h & -2h^3 \\ 0 & 0 & -2h \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & h \\ -2h & 1 & -3h^2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.6.4}$$

PROOF. As far as obtaining this calculus from the $GL_{h,g}(2)$ calculi is concerned, we need only observe that starting with the 4×4 $ABCD$ matrices of Γ_3^{4D}, say, in the relations (4.4.8) and setting $\theta_4 = -\theta_1 - 2h\theta_3$, we obtain commutation relations now involving the 3×3 matrices quoted. That this is the unique 3-dimensional calculus on $SL_h(2)$ follows since these are precisely the $ABCD$ matrices we obtain when we apply the procedure of Section 4 to look for the most general possible 3-dimensional calculus on $SL_h(2)$. $\square$

Now, just as we did for the 4D calculi, we may deduce wedge product commutation relations in the exterior bicovariant graded algebra Ω^{3D}, the Lie brackets and commutators for the quantum Lie algebra $\mathcal{L}^{3D}$, and the enveloping algebra relations for $U(\mathcal{L}^{3D})$. These results are obtained in just the same way as the corresponding results for the 4D calculi so we will not comment on their proofs. We should mention that some results in this direction have already been obtained in **[59]**, where the first order differential calculus on $SL_h(2)$ was postulated through the R-matrix expression (4.5.11). However the results we will present here are perhaps more complete than the corresponding results in **[59]**. Also, it will be useful to collect the results here in our current notation.

THEOREM 4.6.2. *The commutation relations between the left-invariant forms in the external bicovariant graded algebra, Ω^{3D}, built upon the first order calculus Γ^{3D}, are*

$$\begin{aligned}
&\theta_3 \wedge \theta_3 = 0,\\
&\theta_3 \wedge \theta_1 = -\theta_1 \wedge \theta_3,\\
&\theta_3 \wedge \theta_2 = -\theta_2 \wedge \theta_3 + 4h\theta_1 \wedge \theta_3,\\
&\theta_1 \wedge \theta_1 = 0,\\
&\theta_1 \wedge \theta_2 = -\theta_2 \wedge \theta_1 - 2h\theta_2 \wedge \theta_3,\\
&\theta_2 \wedge \theta_2 = 4h\theta_2 \wedge \theta_1 + 8h^2\theta_2 \wedge \theta_3.
\end{aligned} \tag{4.6.5}$$

The relations are such that $\{\theta_2^\alpha\theta_1^\beta\theta_3^\gamma : \alpha, \beta, \gamma \in \{0,1\}\}$ is a basis for the exterior algebra of forms.

THEOREM 4.6.3. *The quantum Lie brackets and commutators for the quantum Lie algebra $\mathcal{L}^{3D}$ are respectively*

$$\begin{aligned}
&[\chi_1, \chi_1] = 0,\\
&[\chi_1, \chi_2] = 2\chi_2,\\
&[\chi_1, \chi_3] = -2\chi_3,\\
&[\chi_2, \chi_1] = -2\chi_2,\\
&[\chi_2, \chi_2] = 0,\\
&[\chi_2, \chi_3] = \chi_1 - 4h\chi_2,\\
&[\chi_3, \chi_1] = -4h\chi_1 + 2\chi_3,\\
&[\chi_3, \chi_2] = -\chi_1,\\
&[\chi_3, \chi_3] = -4h\chi_3,
\end{aligned} \tag{4.6.6}$$

and

$$
\begin{aligned}
[\chi_1, \chi_1] &= 0,\\
[\chi_1, \chi_2] &= \chi_1\chi_2 - \Big(\chi_2\chi_1 + 4h\chi_2\chi_2\Big),\\
[\chi_1, \chi_3] &= \chi_1\chi_3 - \Big(\chi_3\chi_1 - 4h\chi_3\chi_2\Big),\\
[\chi_2, \chi_1] &= \chi_2\chi_1 - \Big(\chi_1\chi_2 - 4h\chi_2\chi_2\Big),\\
[\chi_2, \chi_2] &= 0,\\
[\chi_2, \chi_3] &= \chi_2\chi_3 - \Big(2h\chi_1\chi_2 - 8h^2\chi_2\chi_2 + \chi_3\chi_2\Big),\\
[\chi_3, \chi_1] &= \chi_3\chi_1 - \Big(\chi_1\chi_3 - 8h^2\chi_2\chi_1 + 4h\chi_2\chi_3\Big),\\
[\chi_3, \chi_2] &= \chi_3\chi_2 - \Big(-2h\chi_2\chi_1 + \chi_2\chi_3\Big),\\
[\chi_3, \chi_3] &= \chi_3\chi_3 - \Big(-2h\chi_1\chi_3 + 2h\chi_3\chi_1 - 8h^2\chi_3\chi_2 + \chi_3\chi_3\Big).
\end{aligned}
\tag{4.6.7}
$$

THEOREM 4.6.4. *The relations in the enveloping algebra, $U(\mathcal{L}^{3\mathrm{D}})$, are*

$$
\begin{aligned}
\chi_3\chi_1 &= -8h^2\chi_2\chi_1 + 4h\chi_2\chi_3 + \chi_1\chi_3 - 4h\chi_1 + 2\chi_3,\\
\chi_3\chi_2 &= -2h\chi_2\chi_1 + \chi_2\chi_3 - \chi_1,\\
\chi_1\chi_2 &= 4h\chi_2^2 + \chi_2\chi_1 + 2\chi_2.
\end{aligned}
\tag{4.6.8}
$$

These relations are such that $\{\chi_2^\alpha\chi_1^\beta\chi_3^\gamma : \alpha, \beta, \gamma \in \mathbb{Z}_{\geq 0}\}$ is a basis for $U(\mathcal{L}^{3\mathrm{D}})$.

4.7. The Jordanian quantised universal enveloping algebra

To this point we have been working only with the Jordanian quantum analog of the coordinate ring of $SL_2(\mathbb{C})$. In the remainder of the paper we focus attention on the corresponding deformation of the universal enveloping algebra $U(\mathfrak{sl}_2(\mathbb{C}))$. Let us recall its definition.

DEFINITION 4.7.1. The *Jordanian quantised universal enveloping algebra, $U_h(\mathfrak{sl}_2(\mathbb{C}))$,* is the unital associative algebra over $\mathbb{C}[[h]]$ with generators X, Y, H and relations

$$[H, X] = 2\frac{\sinh hX}{h}, \tag{4.7.1}$$

$$[H, Y] = -Y(\cosh hX) - (\cosh hX)Y, \tag{4.7.2}$$

$$[X, Y] = H. \tag{4.7.3}$$

having a basis $\{Y^\alpha H^\beta X^\gamma : \alpha, \beta, \gamma \in \mathbb{Z}_{\geq 0}\}$.

In [7] the Casimir element, C, of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ was obtained. in our notation it is

$$C = (Y(\sinh hX) + (\sinh hX)Y) + \frac{1}{4}H^2 + \frac{1}{4}(\sinh hX)^2. \tag{4.7.4}$$

Assuming tensor products to be completed in the h-adic topology, the Hopf structure of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ is defined on the generators as,

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \tag{4.7.5}$$

$$\Delta(Y) = Y \otimes e^{hX} + e^{-hX} \otimes Y, \tag{4.7.6}$$

$$\Delta(H) = H \otimes e^{hX} + e^{-hX} \otimes H, \tag{4.7.7}$$

$$\epsilon(X) = 0, \quad \epsilon(Y) = 0, \quad \epsilon(H) = 0, \tag{4.7.8}$$

$$S(X) = -X, \quad S(Y) = -e^{hX} Y e^{-hX}, \quad S(H) = -e^{hX} H e^{-hX}. \tag{4.7.9}$$

It is clear that the element $u = e^{2hX}$ is such that $S^2(x) = uxu^{-1}$ for all $x \in U_h(\mathfrak{sl}_2(\mathbb{C}))$ and $\Delta(u) = u \otimes u$.

$U_h(\mathfrak{sl}_2(\mathbb{C}))$ has received a good deal of attention recently. In particular, we mention the work of Abdesselam *et al* [**1**] in which a non-linear map was constructed which realises the *algebraic* isomorphism between $U_h(\mathfrak{sl}_2(\mathbb{C}))$ and $U(\mathfrak{sl}_2(\mathbb{C}))$. This map was then used by those authors to build the representation theory of $U_h(\mathfrak{sl}_2(\mathbb{C}))$. As with the standard quantisation of the enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$, the representation theory of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ follows very closely the representation theory of $\mathfrak{sl}_2(\mathbb{C})$. Indeed, the finite dimensional, indecomposable representations of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ are in one-to-one correpondence with the finite dimensional irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$, and can be classified as classically by a non-negative half-integer j. Van der Jeugt [**100**] was able to refine the work of Abdesselam *et al*, obtaining closed form expressions for the action of the generators of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ on the basis vectors of finite dimensional irreducible representations. Before Van der Jeugt's work Aizawa [**3**] had demonstrated that the Clebsch-Gordan series for the decomposition of the tensor product of two indecomposable representations of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ was precisely the classical series modulo the one-to-one correspondence of classical and Jordanian representations. Van der Jeugt obtained a general formula for the Clebsch-Gordan coefficients.

4.8. Jordanian quantum Lie algebra from an ad-submodule in $U_h(\mathfrak{sl}_2(\mathbb{C}))$

In the following theorem we describe a left ad-submodule of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ which allows us to build a quantum Lie algebra from the enveloping algebra generators.

THEOREM 4.8.1. *In $U_h(\mathfrak{sl}_2(\mathbb{C}))$ the space spanned by the elements X_h, H_h and Y_h defined by*

$$\begin{aligned} X_h &= e^{hX}\frac{\sinh hX}{h}, \\ H_h &= He^{hX}, \\ Y_h &= Ye^{hX} - 2hC, \end{aligned} \tag{4.8.1}$$

is stable under the left adjoint action of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ on $U_h(\mathfrak{sl}_2(\mathbb{C}))$.

PROOF. To obtain this result, essential use was made of the known PBW basis. With such a basis we can use the computer algebra package REDUCE to perform algebraic manipulations which would be virtually impossible otherwise. In particular we obtain the following actions of the $U_h(\mathfrak{sl}_2(\mathbb{C}))$ generators on the elements $\{X_h, H_h, Y_h\}$ describing a deformation of the adjoint representation of $U(\mathfrak{sl}_2(\mathbb{C}))$,

$$\begin{aligned}
&Y \triangleright X_h = -H_h + 2hX_h, && H \triangleright X_h = 2X_h, && X \triangleright X_h = 0,\\
&Y \triangleright H_h = 2Y_h + 3h^2X_h, && H \triangleright H_h = 4hX_h, && X \triangleright H_h = -2X_h,\\
&Y \triangleright Y_h = -2hY_h - h^2H_h - h^3X_h, && H \triangleright Y_h = -2Y_h - 2hH_h - h^2X_h, && X \triangleright Y_h = H_h.
\end{aligned} \tag{4.8.2}$$

□

The actions of the elements on each other leads to the following *Jordanian quantum Lie brackets* between the elements of the *Jordanian quantum Lie algebra* $\mathfrak{L}_h(\mathfrak{sl}_2(\mathbb{C}))$,

$$\begin{aligned}
&[X_h, Y_h] = H_h - 2hX_h, && [X_h, H_h] = -2X_h, && [X_h, X_h] = 0,\\
&[H_h, Y_h] = -2Y_h - 2hH_h + h^2X_h, && [H_h, H_h] = 0, && [H_h, X_h] = 2X_h\\
&[Y_h, H_h] = 2Y_h - 2hH_h - h^2X_h, && [Y_h, X_h] = -H_h - 2hX_h, && [Y_h, Y_h] = -4hY_h,
\end{aligned} \tag{4.8.3}$$

which display the characteristic h-antisymmetry [**24**]. The $U_h(\mathfrak{sl}_2(\mathbb{C}))$ coproduct on the elements of $\mathfrak{L}_h(\mathfrak{sl}_2(\mathbb{C}))$ is,

$$\begin{aligned}
\Delta(X_h) &= 1 \otimes X_h + X_h \otimes e^{2hX},\\
\Delta(H_h) &= 1 \otimes H_h + H_h \otimes e^{2hX},\\
\Delta(Y_h) &= 1 \otimes Y_h + Y_h \otimes e^{2hX} + 2h(1 \otimes C + C \otimes e^{2hX} - \Delta(C)).
\end{aligned} \tag{4.8.4}$$

A standard definition for the quantum Killing form is the following [**25**].

DEFINITION 4.8.2. The *quantum Killing form* is the map $\mathfrak{B} : \mathfrak{L}_h(\mathfrak{sl}_2(\mathbb{C})) \otimes \mathfrak{L}_h(\mathfrak{sl}_2(\mathbb{C})) \to \mathbb{C}[[h]]$ given by

$$\mathfrak{B}(x, y) = \mathrm{Tr}(xyu) \tag{4.8.5}$$

where the trace Tr is taken over the deformed adjoint representation of $U_h(\mathfrak{sl}_2(\mathbb{C}))$ and u is as defined above.

As explained in [**25**], the Killing form so defined is ad-invariant, non-degenerate, bilinear and satisfies the following simple generalisation of the usual symmetry property,

$$\mathfrak{B}(x, y) = \mathfrak{B}(y, S^2(x)). \tag{4.8.6}$$

Note that,

$$\begin{aligned}
S^2(X_h) &= X_h,\\
S^2(H_h) &= H_h - 4hX_h,\\
S^2(Y_h) &= Y_h + 2hH_h - 4h^2X_h.
\end{aligned} \tag{4.8.7}$$

so that $S^2 : \mathfrak{L}_h(\mathfrak{sl}_2(\mathbb{C})) \to \mathfrak{L}_h(\mathfrak{sl}_2(\mathbb{C}))$.

From the definition it is straightforward to obtain the following evaluations of the quantum Killing form on $\mathfrak{L}_h(\mathfrak{sl}_2(\mathbb{C}))$

$$\begin{aligned} &\mathfrak{B}(H_h, H_h) = 8, && \mathfrak{B}(H_h, X_h) = 0, && \mathfrak{B}(H_h, Y_h) = -8h, \\ &\mathfrak{B}(X_h, H_h) = 0, && \mathfrak{B}(X_h, X_h) = 0, && \mathfrak{B}(X_h, Y_h) = 4, \\ &\mathfrak{B}(Y_h, H_h) = 8h, && \mathfrak{B}(Y_h, X_h) = 4, && \mathfrak{B}(Y_h, Y_h) = -6h^2, \end{aligned} \tag{4.8.8}$$

a simple deformation of the classical Killing form, recovered by setting $h = 0$.

4.9. Jordanian quantum Lie algebra from inverse Clebsch-Gordan coefficients

Classically, for finite dimensional highest weight modules V_Λ and $V_{\Lambda'}$ of a complex simple Lie algebra $\mathfrak{g}$, there is a decomposition

$$V_\Lambda \otimes V_{\Lambda'} \cong \oplus \mathcal{L}^{\Lambda_i}_{\Lambda\Lambda'} V_{\Lambda_i}, \tag{4.9.1}$$

of the tensor product module into a direct sum of its irreducible submodules V_{Λ_i}, the non-negative integers $\mathcal{L}^{\Lambda_i}_{\Lambda\Lambda'}$ being the Littlewood-Richardson coefficients of $\mathfrak{g}$. The dimension of the space of intertwiners between $V_\Lambda \otimes V_{\Lambda'}$ and V_{Λ_i} is then just $\mathcal{L}^{\Lambda_i}_{\Lambda\Lambda'}$. In particular for a pair of adjoint representations, ad, of $\mathfrak{sl}_2(\mathbb{C})$ we have the decomposition

$$\mathrm{ad} \otimes \mathrm{ad} \cong W \oplus \mathrm{ad} \oplus \epsilon, \tag{4.9.2}$$

where ϵ is the trivial (1 dimensional) representation and W is the irreducible representation of dimension 5. Therefore, up to rescaling, there is a unique intertwiner from $\mathrm{ad} \otimes \mathrm{ad} \to \mathrm{ad}$ and a unique intertwiner from $\mathrm{ad} \otimes \mathrm{ad} \to \epsilon$. Indeed, these are precisely the Lie bracket and Killing form of $\mathfrak{sl}_2(\mathbb{C})$ respectively. Choosing bases for the modules W, ad and ϵ, these intertwiners may then be described explicitly by particular subsets of the inverse Clebsch-Gordon coefficients corresponding to the isomorphism (4.9.2).

We know that the representations of $U(\mathfrak{sl}_2(\mathbb{C}))$ and $U_h(\mathfrak{sl}_2(\mathbb{C}))$ are in one-to-one correspondence and that the respective tensor structures of these representations are identical modulo this correspondence. Therefore 'quantised' versions of the Lie bracket and Killing form may be obtained from the intertwiners of the corresponding deformed modules. Moreover, by standard theoretical arguments **[25]** these quantum Lie brackets and Killing form should be identical to those obtained in the last section, up to rescaling. We therefore have a convenient check on the results presented there.

Let us adopt the notation of Van der Jeugt **[100]**. On the representation space $V_h^{(j)}$ with basis e^j_m where j in a non-negative half-integer and $m = -j, -j+1, \dots, j$, the action of the generators X, H and Y is given by [**100**],

$$
\begin{aligned}
H \triangleright e^j_m &= 2m e^j_m, \\
X \triangleright e^j_m &= \sum_{k=0}^{\lfloor (j-m-1)/2 \rfloor} \frac{(h/2)^{2k}}{2k+1} \frac{\alpha_{j,m+1+2k}}{\alpha_{j,m}} e^j_{m+1+2k}, \\
Y \triangleright e^j_m &= (j+m)(j-m+1) \frac{\alpha_{j,m-1}}{\alpha_{j,m}} e^j_{m-1} \\
&\quad - (j-m)(j+m+1) \left(\frac{h}{2}\right)^2 \frac{\alpha_{j,m+1}}{\alpha_{j,m}} e^j_{m+1} \\
&\quad + \sum_{s=1}^{\lfloor (j-m+1)/2 \rfloor} \left(\frac{h}{2}\right)^{2s} \frac{\alpha_{j,m-1+2s}}{\alpha_{j,m}} e^j_{m-1+2s},
\end{aligned}
\tag{4.9.3}
$$

where $\alpha_{j,m} = \sqrt{(j+m)!/(j-m)!}$. Thus the representation matrices of the generators in the deformation of the classical adjoint, $j=1$, representation are,

$$
\begin{aligned}
\Gamma(X) &= \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma(H) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \\
\Gamma(Y) &= \begin{pmatrix} 0 & -\sqrt{2}(h/2)^2 & 0 \\ \sqrt{2} & 0 & -\sqrt{2}(h/2)^2 \\ 0 & \sqrt{2} & 0 \end{pmatrix}.
\end{aligned}
\tag{4.9.4}
$$

The Clebsch-Gordan series for the tensor product of two V^1_h representations is,

$$
V^1_h \otimes V^1_h \cong V^2_h \oplus V^1_h \oplus V^0_h. \tag{4.9.5}
$$

If we denote by v_i the i-th vector in the ordered basis $\{e^2_2, e^2_1, e^2_0, e^2_{-1}, e^2_{-2}, e^1_1, e^1_0, e^1_{-1}, e^0_0\}$ and by w_i the i-th vector in the ordered basis $\{e^1_1 \otimes e^1_1, e^1_1 \otimes e^1_0, e^1_1 \otimes e^1_{-1}, e^1_0 \otimes e^1_1, e^1_0 \otimes e^1_0, e^1_0 \otimes e^1_{-1}, e^1_{-1} \otimes e^1_1, e^1_{-1} \otimes e^1_0, e^1_{-1} \otimes e^1_{-1}\}$, then the Clebsch-Gordan matrix C, where $v_i = \sum_{j=1}^9 C_{ij} w_j$ is given by [**100**]

$$
C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\
\frac{\sqrt{2}h^2}{2\sqrt{3}} & \frac{-h}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{h}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \\
0 & \frac{\sqrt{2}h^2}{2} & -h & \frac{\sqrt{2}h^2}{2} & 0 & \frac{1}{\sqrt{2}} & h & \frac{1}{\sqrt{2}} & 0 \\
\frac{-h^4}{4} & \frac{-\sqrt{2}h^3}{2} & \frac{3h^2}{2} & \frac{\sqrt{2}h^3}{2} & 0 & -\sqrt{2}h & \frac{3h^2}{2} & \sqrt{2}h & 1 \\
-2h & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & -h & \frac{1}{\sqrt{2}} & -h & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \\
0 & \frac{\sqrt{2}h^2}{2} & -h & \frac{-\sqrt{2}h^2}{2} & 0 & \frac{1}{\sqrt{2}} & -h & \frac{-1}{\sqrt{2}} & 0 \\
\frac{h^2}{\sqrt{3}} & \frac{-\sqrt{2}h}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{\sqrt{2}h}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0
\end{pmatrix}, \tag{4.9.6}
$$

with inverse

$$C^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}h & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ \frac{3h^2}{2} & h & \frac{\sqrt{6}}{6} & 0 & 0 & h & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{3}}{3} \\ -\sqrt{2}h & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & \frac{-\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{6}}{3} & 0 & 0 & 0 & 0 & 0 & \frac{-\sqrt{3}}{3} \\ \frac{\sqrt{2}h^3}{2} & \frac{\sqrt{2}h^2}{2} & \frac{\sqrt{3}h}{3} & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}h^2}{2} & h & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}h}{3} \\ \frac{3h^2}{2} & -h & \frac{\sqrt{6}}{6} & 0 & 0 & h & \frac{-\sqrt{2}}{2} & 0 & \frac{\sqrt{3}}{3} \\ \frac{-\sqrt{2}h^3}{2} & \frac{\sqrt{2}h^2}{2} & \frac{-\sqrt{3}h}{3} & \frac{\sqrt{2}}{2} & 0 & \frac{-\sqrt{2}h^2}{2} & h & \frac{-\sqrt{2}}{2} & \frac{-\sqrt{6}h}{3} \\ \frac{-h^4}{4} & 0 & \frac{\sqrt{6}h^2}{6} & 0 & 1 & 0 & 0 & 2h & \frac{\sqrt{3}h^2}{3} \end{pmatrix}. \tag{4.9.7}$$

Considering $w_i = \sum_{j=1}^{9} C_{ij}^{-1} v_j$ we see that columns 6–8 of C^{-1} correspond to an intertwiner $\mathrm{ad} \otimes \mathrm{ad} \to \mathrm{ad}$ and we deduce a quantum Lie bracket on the vectors $\{e_1^1, e_0^1, e_{-1}^1\}$,

$$\begin{aligned} &[e_1^1, e_{-1}^1] = \frac{\sqrt{2}}{2} e_0^1 + h e_1^1, && [e_1^1, e_0^1] = \frac{\sqrt{2}}{2} e_1^1, && [e_1^1, e_1^1] = 0, \\ &[e_0^1, e_{-1}^1] = \frac{\sqrt{2}}{2} e_{-1}^1 + h e_0^1 + \frac{\sqrt{2}}{2} h^2 e_1^1, && [e_0^1, e_0^1] = 0, && [e_0^1, e_1^1] = -\frac{\sqrt{2}}{2} e_1^1, \\ &[e_{-1}^1, e_0^1] = -\frac{\sqrt{2}}{2} e_{-1}^1 + h e_0^1 - \frac{\sqrt{2}}{2} h^2 e_1^1, && [e_{-1}^1, e_{-1}^1] = 2h e_{-1}^1, && [e_{-1}^1, e_1^1] = -\frac{\sqrt{2}}{2} e_0^1 + h e_1^1. \end{aligned} \tag{4.9.8}$$

Similarly, column 9 of C^{-1} corresponds to an intertwiner $\mathrm{ad} \otimes \mathrm{ad} \to \mathbb{C}[[h]]$ and we obtain the Killing form

$$\begin{aligned} &\mathfrak{B}(e_0^1, e_0^1) = -\frac{\sqrt{3}}{3}, && \mathfrak{B}(e_0^1, e_1^1) = 0, && \mathfrak{B}(e_0^1, e_{-1}^1) = \frac{\sqrt{6}h}{3}, \\ &\mathfrak{B}(e_1^1, e_0^1) = 0, && \mathfrak{B}(e_1^1, e_1^1) = 0, && \mathfrak{B}(e_1^1, e_{-1}^1) = \frac{\sqrt{3}}{3}, \\ &\mathfrak{B}(e_{-1}^1, e_0^1) = -\frac{\sqrt{6}h}{3}, && \mathfrak{B}(e_{-1}^1, e_1^1) = \frac{\sqrt{3}}{3}, && \mathfrak{B}(e_{-1}^1, e_{-1}^1) = \frac{\sqrt{3}h^2}{3}, \end{aligned} \tag{4.9.9}$$

Now, if we perform the following change of basis,

$$\begin{aligned} X_h &= 2e_1^1, \\ H_h &= 4he_1^1 - 2\sqrt{2}e_0^1, \\ Y_h &= -\frac{5}{2}h^2 e_1^1 + 2\sqrt{2}he_0^1 - 2e_{-1}^1, \end{aligned} \tag{4.9.10}$$

then the representation matrices of the generators become,

$$\Gamma(X) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma(H) = \begin{pmatrix} 2 & 4h & -h^2 \\ 0 & 0 & -2h \\ 0 & 0 & -2 \end{pmatrix},$$

$$\Gamma(Y) = \begin{pmatrix} 2h & 3h^2 & -h^3 \\ -1 & 0 & -h^2 \\ 0 & 2 & -2h \end{pmatrix}, \tag{4.9.11}$$

which are precisely those obtained from (4.8.2) and the Lie bracket relations become (those already obtained in 4.8.3),

$$
\begin{aligned}
&[X_h, Y_h] = H_h - 2hX_h, && [X_h, X_h] = 0, && [X_h, H_h] = -2X_h, \\
&[H_h, Y_h] = -2Y_h - 2hH_h + h^2X_h, && [H_h, H_h] = 0, && [H_h, X_h] = 2X_h, \\
&[Y_h, H_h] = 2Y_h - 2hH_h - h^2X_h, && [Y_h, X_h] = -H_h - 2hX_h, && [Y_h, Y_h] = -4hY_h.
\end{aligned} \tag{4.9.12}
$$

Further, we can scale the Killing form, by scaling the single basis vector of V_h^0 so that on the $\{X_h, H_h, Y_h\}$ it reads

$$
\begin{aligned}
&\mathfrak{B}(H_h, H_h) = 8, && \mathfrak{B}(H_h, X_h) = 0, && \mathfrak{B}(H_h, Y_h) = -8h, \\
&\mathfrak{B}(X_h, H_h) = 0, && \mathfrak{B}(X_h, X_h) = 0, && \mathfrak{B}(X_h, Y_h) = 4, \\
&\mathfrak{B}(Y_h, H_h) = 8h, && \mathfrak{B}(Y_h, X_h) = 4, && \mathfrak{B}(Y_h, Y_h) = -6h^2,
\end{aligned} \tag{4.9.13}
$$

precisely as was found above.

4.10. Conclusion

Returning to the quantum Lie algebra obtained through Woronowicz's bicovariant calculus, $\mathcal{L}^{3D}$, if we change the basis according to the identifications,

$$
\begin{aligned}
H_h &= \chi_1, \\
X_h &= \chi_2, \\
Y_h &= -h\chi_1 + \frac{h^2}{4}\chi_2 + \chi_3
\end{aligned} \tag{4.10.1}
$$

then the *Woronowicz quantum Lie bracket* on these new basis elements is *precisely* that already found in (4.8.3). Thus, as algebras over $\mathbb{C}[[h]]$, the Woronowicz and 'Sudbery-Delius' quantum Lie algebras are *isomorphic*. This means, furthermore, that in addition to having a Killing form we have some natural analog of the Jacobi identity for this Jordanian quantum lie algebra.

We had already found two appealing aspects of the bicovariant differential geometry on $SL_h(2)$. Namely, its uniqueness and 3-dimensionality. The fact that the Woronowicz quantum Lie algebra is isomorphic to the Sudbery-Delius quantum Lie algebra we found starting with $U_h(\mathfrak{sl}_2(\mathbb{C}))$ is a further attractive feature. Recently, the work of Aghamohammadi [2] and Cho, Madore and Park [18] have shown that the corresponding Jordanian quantum plane admits a richer geometrical structure than the standard quantum plane. It should be interesting to try to develop further the geometry on the Jordanian quantum group and also investigate possible $SL(n)$ generalisations.

Bibliography

[1] B. Abdesselam, A. Chakrabarti and R. Chakrabarti, *Irreducible representations of Jordanian quantum algebra $U_h(sl(2))$ via a nonlinear map,* Mod. Phys. Lett. A, **11** (1996), 2883–2891.

[2] A. Aghamohammadi, *$SL_h(2)$-symmetric torsionless connections,* Lett. Math. Phys. **40** (1997), 95.

[3] N. Aizawa, preprint q-alg/9701022

[4] B. L. Aneva, V. K. Dobrev and S. G. Mihov, *Duality for the Jordanian Matrix Quantum Group $GL_{g,h}(2)$,* INRNE preprint, INRNE-TH/3/97, q-alg/ (1997).

[5] G. E. Arutuynov, A. P. Isaev and Z. Popowicz, *Poincaré-Bitkhoff-Witt property for bicovariant algebras on simple quantum groups,* Steklov preprint, q-alg/9502002 (1995).

[6] M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra,* Addison-Wesley, Reading, Mass., 1969.

[7] A. Ballesteros, F. J. Herranz, M. A. del Olmo and M. Santander, J. Phys. A, **28** (1995), 7113.

[8] J. Balog, L. Dabrowski and L. Feher, *Non-standard quantum group in Toda and WZNW theories,* in Nonperturbative methods in low dimensional quantum field theories, World Scientific, 279–293 (1991).

[9] F. Bayen, M. Flato, M, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, *Deformation theory and quantisation I. Deformations of symplectic structures,* Ann. Phys. **111** (1978), 61–110.

[10] A. A. Belavin and V. G. Drinfeld, *Triangle equations and simple Lie algebras,* in Mathematical Physics Reviews, edited by S. P. Novikov, Harwood, New York, 93–166 (1984).

[11] G. Bergman, *The diamond lemma for ring theory,* Advances in Mathematics **29** (1978), 178–218.

[12] M. Bremner, *Quantum deformations of simple Lie algebras,* Canad. Math. Bull. **40** (1997), 143–148.

[13] K. Bresser, *All bicovariant differential calculi on $GL_q(3,\mathbb{C})$ and $SL_q(3,\mathbb{C})$,* J. Phys. A Math Gen., **28** (1995), 2545–2562.

[14] T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups,* Graduate Texts in Mathematics, vol. 98, Springer-Verlag, New York, 1985.

[15] T. Brzeziński, *Remarks on bicovariant differential calculi and exterior Hopf algebras,* Lett. Math. Phys., **27** (1993), 287.

[16] R. Carter, G. Segal and I. MacDonald, *Lectures on Lie Groups and Lie algebras,* London Mathematical Society Student Texts, vol. 32, Cambridge University Press, Cambridge, 1995.

[17] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge (1994).

[18] S. Cho, J. Madore and K. S. Park, *Noncommutative Geometry of the h-deformed Quantum Plane,* Semyung preprint, SMHEP 97/8, q-alg/9709007 (1997).

[19] A. Connes, *Non-Commutative Differential Geometry,* Publications of Inst. des Hautes Etudes Scientifiques, **62** (1986), 257.

[20] A. Connes, *Non-Commutative Geometry,* Academic Press, 1994.

[21] J. F. Cornwell, *Group Theory in Physics, Volume 2,* Techniques in Physics, vol. 7, Academic Press, London, 1984.

[22] E. Cremmer and J.-L. Gervais, *The quantum group structure associated with non-linearly extended Virasoro algebras,* Comm. Math. Phys., **134**, 619–632 (1990).

[23] G. W. Delius and A. Hüffmann, *On quantum Lie algebras and quantum root systems,* J. Phys. A, **29** (1996), 1703.

[24] G. W. Delius and M. D. Gould, *Quantum Lie algebras, their existence, uniqueness and q-antisymmetry,* Commun. Math. Phys. **185** (1997), 709–722.

[25] G. W. Delius, C. Gardner and M. D. Gould, *The structure of quantum Lie algebras for the classical series B_l, C_l and D_l,* KCL preprint, q-alg/9706029 (1996).

[26] E. E. Demidov, Yu. I. Manin, E. E. Mukhin and D. V. Zhdanovich, *Non-standard quantum deformations of GL(n) and constant solutions of the Yang-Baxter equation,* Progr. Theor. Phys. Suppl., **102** (1990), 203–218.

[27] E. E. Demidov, *Some aspects of the theory of quantum groups,* Russian Math. Surveys, **48** (1993), 41–79.

[28] M. S. Dijkhuizen, *On Compact Quantum Groups and Homogeneous Spaces,* PhD thesis, University of Amsterdam, Amsterdam (1994).

[29] Y. Doi, *Braided bialgebras and quadratic bialgebras,* Comm. Alg. **21**, 1731–1749 (1993).

[30] Y. Doi and M. Takeuchi, *Multiplication alteration by two-cocycles - the quantum version,* Comm. Alg., **22**, 5715–5732 (1994).

[31] J. Donin and D. Gurevich, *Braiding of the Lie algebra* $\mathfrak{sl}_2(\mathbb{C})$, Amer. Math. Soc. Transl. (2) **167** (1995), 23–26.

[32] V. G. Drinfeld, *Quantum groups,* in Proceedings of the International Congress of Mathematicians, Berkeley, 1986, edited by A. M. Gleason, American Mathematical Society, Providence, 798–820 (1987)

[33] V. G. Drinfeld, *Quasi-Hopf algebras,* Leningrad Math. J., **1**, 1419–1457 (1990).

[34] V. G. Drinfeld, *On some unsolved problems in quantum group theory,* in Quantum Groups, Proceedings of Workshops held in the Euler International Mathematical Institute, 1990, edited by P. P. Kulish, Lect. Notes Math. **1510**, 1–8 (1992).

[35] V. G. Drinfeld, *On quasitriangular quasi-Hopf algebras and a group closely connected with* $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J. **2**, 829–860 (1991).

[36] V. G. Drinfeld, *On the structure of quasitriangular quasi-Hopf algebras,* Funct. Anal. Appl. **26**, 63–65 (1992).

[37] R. A. Engeldinger and A. Kempf, *Sub-Hopf-algebra-induced twists of quantum enveloping algebras,* J. Math. Phys. **35**, 1931–1938 (1994).

[38] P. Etingof and D. Kazhdan, *Quantisation of Lie bialgebras, I,* preprint, Harvard University, (1996).

[39] L. D. Faddeev and P. N. Pyatov, *The Differential Calculus on Quantum Linear Groups,* Amer. Math. Soc. Transl. **175** (1996) 35–47.

[40] H. Flanders, *Differential Forms with Applications to the Physical Sciences,* Dover Publications, New York (1989).

[41] C. Fronsdal and A. Galindo, *The Universal* T*-matrix,* Contemp. Math., **175**, 73–88 (1994).

[42] C. Fronsdal and A. Galindo, *Deformations of Multiparameter Quantum* $gl(n)$, Lett. Math. Phys., **34**, 25–36 (1995).

[43] W. Fulton and J. Harris, *Representation Theory, A First Course,* Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991.

[44] M. Gerstenhaber, A. Giaquinto and S. D. Schack, *Construction of quantum groups from Belavin-Drinfeld infinitesimals,* in Quantum Deformations of Algebras and their Representations, edited by A. Joseph and S. Shnider, Israel Math. Conf. Series 7, Bar-Ilan Univ., 45–64 (1993).

[45] A. Guichardet, *On the Definitions of the Quantum Group* $U_h(sl(2,k))$ *and the Restricted Dual of* $U_h(sl(n,k))$, in Noncompact Lie Groups and Some of Their Applications, edited by E. A. Tanner and R. Wilson, Kluwer Academic Publishers, Netherlands, 413–422 (1994).

[46] A. Guichardet, *Groupes quantiques, Introduction au point de vue formel,* Intereditions, CNRS, Paris (1995).

[47] D. I. Gurevich, *Hecke symmetries and quantum determinants,* Dokl. Akad. Nauk. SSSR, **303** (1988), 542–546 (Russian); English transl. in Soviet Math. Dokl., **38** (1989), 555–559.

[48] A. C. Hearn and J. P. Fitch (ed.), *REDUCE User's Manual 3.6,* Konrad-Zuse-Zentrum für Informationstechnik Berlin, Berlin (1995).

[49] I. Heckenberger and K. Schmüdgen, *Classification of Bicovariant Differential Calculi on the Quantum Groups* $SL_q(n+1)$ *and* $Sp_q(2n)$, Leipzig preprint (1997), q-alg/

[50] T. J. Hodges, *Nonstandard quantum groups associated to certain Belavin-Drinfeld triples,* Contemp. Math., to appear.

[51] T. J. Hodges, T. Levasseur and M. Toro, *Algebraic structure of multi-parameter quantum groups,* Advances in Math., to appear.

[52] T. J. Hodges, *On the Cremmer-Gervais quantisations of* $SL(n)$, Internat. Math. Res. Notices, **10**, 465–481 (1995).

[53] T. J. Hodges, *Double quantum groups and Iwasawa decomposition,* J. Algebra, to appear.

[54] J. C. Jantzen, *Lectures on Quantum Groups,* Graduate Studies in Mathematics, vol. 6, American Mathematical Society, Rhode Island, 1996.

[55] M. Jimbo, *A q-analogue of $U(gl(N+1))$, Hecke algebra and the Yang-Baxter equation,* Lett. Math. Phys. **11**, 247–252 (1986).

[56] B. Jurco, *Differential calculus on quantised simple Lie groups,* Lett. Math. Phys., **22** (1991), 177–186.

[57] B. Jurco, *Differential Calculus on Quantum Groups: Constructive Procedure,* CERN preprint, hep-th/9408179 (1994).

[58] V. Karimipour, *The Quantum De Rham Complexes Associated with $SL_h(2)$,* Lett. Math. Phys., **30** (1994), 87–98.

[59] V. Karimipour, *Bicovariant differential geometry of the quantum group $SL_h(2)$,* Lett. Math. Phys., **35** (1995), 303–311.

[60] C. Kassel, *Quantum Groups,* Graduate Texts in Mathematics, vol. 155, Springer-Verlag, New York, 1995.

[61] A. Kempf, *Multiparameter R-matrices, Sub-Quantum-Groups and Generalised Twisting Method,* in Proceedings of the XXth International Conference on Differential Geometric Methods in Theoretical Physics, World Scientific, Singapore, 546 (1991).

[62] S. M. Khoroshkin and V. N. Tolstoy, *Twisting of quantum (super)algebras. Connection of Drinfeld's and Cartan-Weyl realizations for quantum affine algebras,* Bonn preprint MPI/94-23 (1994).

[63] J. L. Koszul, *Lectures on Fibre Bundles and Differential Geometry,* Tata Institute of Fundamental Research, Bombay.

[64] P. P. Kulish and N. Reshetikhin, *Quantum linear problem for the sine-Gordan equation and higher representations,* Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **101** (1981), 101–110.

[65] R. G. Larson and J. Towber, *Two dual classes of bialgebras related to the concept of quantum group and quantum Lie algebra,* Comm. Alg. **19**, 3295–3345 (1991).

[66] A. Yu. Lazarev and M. V. Movshev, *On the quantisation of certain Lie groups and Lie algebras,* Russ. Math. Surveys, **46:6** (1991), 225–226.

[67] V. Lyubashenko and A. Sudbery, *Quantum Lie algebras of type A_n,* University of York preprint, q-alg/9510004 (1995).

[68] J. Madore, *An Introduction to Noncommutative Differential Geometry and its Physiacl Applications,* Cambridge University Press, Cambridge 1995.

[69] S. Majid, *Tanaka-Krein theorem for quasi-Hopf algebras and other results,* Contemp. Math. **134**, 219–232 (1992).

[70] S. Majid, *Quasitriangular Hopf algebras and Yang-Baxter equations,* Int. J. Mod. Phys. A, **5**, 1–91 (1990).

[71] S. Majid, *Foundations of quantum group theory,* Cambridge University Press, Cambridge (1995).

[72] G. Maltsiniotis, *Calcul différentiel sur le groupe linéaire quantique,* C. R. Acad. Sci. Sér. I, **311** (1990), 831.

[73] Yu. I. Manin, *Quantum Groups and Noncommutative Geometry,* Centre de Recherche Mathématiques, Montréal (1988).

[74] Yu. I. Manin, *Notes on quantum groups and quantum de Rham complexes,* Teor. Mat. Fiz., **92** (1992), 425.

[75] J. E. Moyal, *Quantum mechanics as a statistical theory,* Proc. Camb. Phil. Soc. **45** (1949), 99–124.

[76] E. E. Mukhin, *Yang-Baxter operators and the non-commutative de Rham complex,* Russ. Math. Surveys, **46:4** (1991), 192–193.

[77] F. Müller Hoissen, *Differential calculi on the quantum group $GL_{p,q}(2)$,* J. Phys. A Math. Gen., **25** (1992), 1703–1734.

[78] F. Müller Hoissen, *Noncommutative differential calculus, quantum groups, and computer algebra,* in G. Jacobs *et al.* (eds), Proc. Int. IMACS Symposium on Symbolic Computation, Lab. d'Informatique Fondamentale, Lille (1993).

[79] F. Müller Hoissen, *Differential Calculi on quantum (sub-) groups and their classical limit,* in Proc. Int. Symposium on Generalised Symmetries in Physics, ASI, Clausthal (1993).

[80] F. Müller Hoissen and C. Reuten, *Bicovariant differential calculi on $GL_{p,q}(2)$ and quantum subgroups,* J. Phys. A Math. Gen., **26** (1993), 2955–2975.

[81] C. Ohn, *A $*$-product on SL(2) and the corresponding nonstandard quantum U(sl(2))*, Lett. Math. Phys., **25** (1992), 85–88.

[82] A. L. Onishchik and E. B. Vinberg, *Lie Groups and Algebraic Groups*, Springer Series in Soviet Mathematics, Springer Verlag, Berlin (1990).

[83] N. Y. Reshetikhin, *Multi-parameter quantum groups and twisted quasi-triangular Hopf algebras*, Lett. Math. Phys., **20**, 331–335 (1990).

[84] N. Yu. Reshetikhin, L. A. Takhtajan, and L. D. Faddeev, *Quantisation of Lie groups and Lie algebras*, Leningrad Math. J., **1**, 193–225 (1990).

[85] A. Schirrmacher, *Multiparameter R-matrices and their quantum groups*, J. Phys. A. **24**, L1249–L1258 (1991).

[86] A. Schirrmacher, *Remarks on the use of R-matrices*, in R. Gierlak *et al.* (eds), Quantum Groups and Related Topics, Kluwer, Dordrecht (1992), 55–66.

[87] K. Schmüdgen and A. Schüler, *Covariant differential calculi on quantum spaces and on quantum groups*, C. R. Acad. Sci Paris, **316** (1993), 1155–1160.

[88] K. Schmüdgen and A. Schüler, *Classification of Bicovariant Differential Calculi on Quantum Groups of Type A, B, C and D*, Commun. Math. Phys., **167** (1995), 635–670.

[89] K. Schmüdgen and A. Schüler, *Classification of Bicovariant Differential Calculi on Quantum Groups*, Commun. Math. Phys., **170** (1995), 315–335.

[90] I. Segal, *Review of "Noncommutative Geometry" by Alain Connes*. Bull. Amer. Math. Soc. **33** (1996) 459–465.

[91] I. R. Shafarevich, *Basic Notions of Algebra*, Springer-Verlag, Berlin, 1997.

[92] B. Simon, *Representations of Finite and Compact Groups*, Graduate Studies in Mathematics. vol. 10. American Mathematical Society, Rhode Island, 1996.

[93] E. K. Sklyanin, *On an algebra generated by quadratic relations*. Uspekhi Mat. Nauk **40** (1985). 214. (Russian)

[94] A. Stolin, *On rational solutions of the classical Yang-Baxter equation*, Thesis, Matematiska Institutionen, Stockholms Universitet.

[95] A. Sudbery, *Consistent multi-parameter quantisation of GL(n)*. J. Phys. A, **23**, L697–L704 (1990).

[96] A. Sudbery, *Matrix element bialgebras determined by quadratic co-ordinate algebras*, J. Algebra, **158** (1994), 375–399.

[97] A. Sudbery, *Canonical differential calculus on quantum general linear groups and supergroups*, Phys. Lett. B, **284** (1992), 61–65.

[98] M. Sweedler, *Hopf Algebras*, W. A. Benjamin, Inc. New York (1969)

[99] M. Takeuchi, *A two-parameter quantisation of GL_n (Summary)*, Proc. Japan Acad. Ser. A Math. Sci. **66**, 112–114 (1990).

[100] J. Van der Jeugt, *Representations and Clebsch-Gordan coefficients for the Jordanian quantum algebra $U_h(sl(2))$*, Gent preprint, q-alg/9703011 (1997).

[101] E. B. Vinberg, *Linear Representations of Groups*, Basler Lehrbücher, Birkhäuser Verlag, Basel (1989).

[102] H. Weyl, *Quantenmechanik und Gruppentheorie*, Z. Phys. **46** (1927) 1–46.

[103] S. L. Woronowicz, *Twisted SU(2) Group. An example of a Non-Commutative Differential Calculus*. Publ. RIMS, Kyoto Univ.. **23** (1987), 117.

[104] S. L. Woronowicz, *Differential Calculus on Compact Matrix Pseudogroups (Quantum Groups)*, Commun. Math. Phys., **122** (1989). 125–170.

[105] S. Zakrzewski, *A Hopf star-algebra of polynomials on the quantum SL(2, R) for a "unitary" R-matrix*, Lett. Math. Phys. **22** (1991). 287–298.